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A GENERALIZATION OF GREENBERG'S $\mathcal{L}$-INVARIANT

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Abstract. Using the theory of (φ,Γ)-modules we generalize Greenberg's construction of the $\mathcal{L}$-invariant to p-adic representations which are semistable at p.

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Introduction

0.1. Let $f=\sum\limits_{n=1}^{\infty}a_nq^n$ be a normalized newform of weight $2k$ on $\Gamma_0(Np)$ where $(N,p)=1$ and let $L(f,s)=\sum\limits_{n=1}^{\infty}a_nn^{-s}$ denote the complex L-function associated to f. The Euler factor of $L(f,s)$ at p can be written in the form $E_p(f,p^{-s})$ where $E_p(f,X)=\prod\limits_{i}^{d}(1-\alpha_iX)$ with $d\leqslant 2$. To any $\alpha\in\{\alpha_i \mid 1\leqslant i\leqslant d\}$ such that $v_p(\alpha)<2k-1$ one can associate a p-adic L-function $L_{p,\alpha}(f,s)$ interpolating p-adically the special values $L(f,j)/\Omega_f$ $(1\leqslant j\leqslant 2k-1)$ where Ω_f denotes the Shimura period of f.

Assume that $U_p(f)=p^{k-1}f$ where U_p is the Atkin-Lehner operator. Then $E_p(f,X)=1-p^{k-1}X$ and we denote by $L_p(f,s)$ the p-adic L-function associated to $\alpha=p^{k-1}$. The interpolation property forces $L_p(f,s)$ to vanish at $s=k$. In [MTT] Mazur, Tate and Teitelbaum conjectured that there exists an invariant $\mathcal{L}(f)$ which depends only on the restriction of the p-adic Galois representation V_f attached to f to a decomposition group at p and such that

$$L_p'(f,k)=\mathcal{L}(f)\,\frac{L(f,k)}{\Omega_f}. \tag{1}$$

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In the weight two case f corresponds to an elliptic curve $E/\mathbb{Q}$ having split multiplicative reduction at p. The p-adic representation V_f is ordinary and seats in an exact sequence of the form

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow V_f \longrightarrow \mathbb{Q}_p \longrightarrow 0.$$

The class of this extension in $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ coincides with the image of the Tate invariant q_E under the Kummer homomorphism and Mazur, Tate and Teitelbaum conjectured that $\mathcal{L}(f) = \dfrac{\log_p q_E}{\mathrm{ord}_p q_E}$. In the higher weight case several definitions were proposed [Tm], [Co], [Mr1], [O], [Br]. It is now known that all these invariants are equal. Remark that the Fontaine-Mazur $\mathcal{L}$-invariant [Mr1] is defined in terms of the filtered (φ, N)-module $\mathbf{D}_{\mathrm{st}}(V_f)$ and has a natural interpretation in the theory of (φ,Γ)-modules [Cz4]. The conjecture (1) was first proved by Greenberg and Stevens [GS] in the weight two case. In [S], Stevens generalized this proof to the higher weights. Other proofs were found by Kato, Kurihara and Tsuji, Orton, Emerton, ... and we refer to [Cz3] and [BDI] for further information and references. On the other hand,in [G] Greenberg defined an $\mathcal{L}$-invariant for pseudo-geometric representations which are ordinary at p and suggested a natural generalization of the conjecture(1). Important results in this direction were recently proved by Hida [Hi].

0.2. In this paper we propose a definition of the $\mathcal{L}$-invariant for representations which are semistable at p generalizing the both Fontaine's and Greenberg's constructions. In the subsequent paper [Ben2] is proved that this definition is compatible with Perrin-Riou's theory of p-adic L-functions [PR]. The main technical tool is the theory of (φ,Γ)-modules [F1], [Cz2]. We make use of Colmez's observation that the (φ,Γ)-module associated to an irreducible p-adic representation may be reducible in the category of (φ,Γ)-modules over the Robba ring $\mathcal{R}$ [Cz4]. In particular, semistable representations are trianguline [BC] and we follow Greenberg's approach using the cohomology of (φ,Γ)-modules instead Galois cohomology. In §1, for any (φ,Γ)-module D over $\mathcal{R}$ we define a subgroup $H^1_f(D)$ of $H^1(D)$ which generalizes $H^1_f(\mathbb{Q}_p, V)$ of Bloch and Kato [BK] and transpose some classical properties of these groups to our situation. The proofs are not difficult and follow from fundamental results of Berger [Ber1], [Ber2], but do not seem to be in the literature and we give them in all details. In §2 the $\mathcal{L}$-invariant is defined, and some related results and conjectures are discussed.

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§1. Cohomology of (φ,Γ)-modules

1.1. Preliminaries.

1.1.1. Rings of p-adic periods (see [F2], [Ber1], [Cz2]). Let K be a finite extension of $\mathbb{Q}_p$. We write K_0 for the maximal unramified subextension of K, O_K for the ring of integers of K and k_K for its residue field. Let σ denote the absolute Frobenius of $K_0/\mathbb{Q}_p$. For any perfect ring A we write $W(A)$ for the ring of Witt vectors with coefficients in A. In particular $O_{K_0} = W(k_K)$. Fix an algebraic closure $\overline{K}/K$ and set $G_K = \mathrm{Gal}(\overline{K}/K)$. We denote by C the completion of $\overline{K}$ and write $v_p \ : \ C \to \mathbb{R} \cup \{\infty\}$ for the p-adic valuation normalized so that $v_p(p) = 1$. Set $|x|_p = \left(\frac{1}{p}\right)^{v_p(x)}$. Let μ_{p^n} denote the group of p^n-th roots of unity. Fix a system of primitive roots of unity $\varepsilon = (\zeta_{p^n})_{n \geqslant 0}$, such that $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for all $n \geqslant 1$. Set $K_n = K(\zeta_{p^n})$, $K_\infty = \bigcup_{n=0}^{\infty} K_n$, $H_K = \mathrm{Gal}(\bar{K}/K_\infty)$, $\Gamma = \mathrm{Gal}(K_\infty/K)$ and denote by $\chi \, : \, \Gamma \to \mathbb{Z}_p^*$ the cyclotomic character.

Consider the projective limit

$$\widetilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^p} O_C / \, p \, O_C$$

where O_C is the ring of integers of C. Then $\widetilde{\mathbf{E}}^+$ is the set of $x=(x_0,x_1,\dots,x_n,\dots)$ such that $x_n\in O_C/\,p\,O_C$ and $x_{n+1}^p=x_n$ for all n. For each n choose a lifting $\hat{x}_n\in O_C$ of x_n. Then for all $m\geqslant 0$ the sequence $\hat{x}_{m+n}^{p^n}$ converges to $x^{(m)}=\lim_{n\to\infty}\hat{x}_{m+n}^{p^n}\in O_C$ which does not depend on the choice of $\hat{x}_n$. The ring $\widetilde{\mathbf{E}}^+$ equipped with the valuation $v_{\mathbf{E}}(x)=v_p(x^{(0)})$ is a complete local ring of characteristic p with residue field $\overline{\mathbb{F}}_p$. Moreover it is integrally closed in his field of fractions $\widetilde{\mathbf{E}}=\mathrm{Fr}(\widetilde{\mathbf{E}}^+)$.

Let $\widetilde{\mathbf{A}}=W(\widetilde{\mathbf{E}})$ be the ring of Witt vectors with coefficients in $\widetilde{\mathbf{E}}$. Denote by $[\;]:\widetilde{\mathbf{E}}\to W(\widetilde{\mathbf{E}})$ the Teichmuller lift. Any $u=(u_0,u_1,\dots)\in\widetilde{\mathbf{A}}$ can be written in the form

$$u=\sum_{n=0}^{\infty}[u^{p^{-n}}]p^n.$$

Set $\pi=[\varepsilon]-1$, $\mathbf{A}_{K_0}^+=O_{K_0}[[\pi]]$ and denote by $\mathbf{A}_{K_0}$ the p-adic completion of $\mathbf{A}_{K_0}^+[1/\pi]$. This is a discrete valuation ring with residue field $\mathbf{E}_{K_0}=k_K((\varepsilon-1))$. Let $\widetilde{\mathbf{B}}=\widetilde{\mathbf{A}}\,[1/p]$ and $\mathbf{B}_{K_0}=\mathbf{A}_{K_0}\,[1/p]$ and let $\mathbf{B}$ denote the completion of the maximal unramified extension of $\mathbf{B}_{K_0}$ in $\widetilde{\mathbf{B}}$. Set $\mathbf{A}=\mathbf{B}\cap\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{A}}^+=W(\mathbf{E}^+)$, $\mathbf{A}^+=\widetilde{\mathbf{A}}^+\cap\mathbf{A}$ and $\mathbf{B}^+=\mathbf{A}^+\,[1/p]$. All these rings are endowed with natural actions of the Galois group G_K and Frobenius φ.

Set $\mathbf{A}_K=\mathbf{A}^{H_K}$ and $\mathbf{B}_K=\mathbf{A}_K\,[1/p]$. Remark that Γ and φ act on $\mathbf{B}_{K_0}$ by

$$\begin{aligned}\tau(\pi)&=(1+\pi)^{\chi(\tau)}-1, \qquad \tau\in\Gamma\\ \varphi(\pi)&=(1+\pi)^p-1.\end{aligned}$$

For any $r>0$ define

$$\widetilde{\mathbf{B}}^{\dagger,r}\;=\;\left\{x\in\widetilde{\mathbf{B}}\mid\lim_{k\to+\infty}\left(v_{\mathbf{E}}(x_k)\;+\;\frac{pr}{p-1}\,k\right)\;=\;+\infty\right\}.$$

Set $\mathbf{B}^{\dagger,r}=\mathbf{B}\cap\widetilde{\mathbf{B}}^{\dagger,r}$, $\mathbf{B}_K^{\dagger,r}=\mathbf{B}_K\cap\mathbf{B}^{\dagger,r}$, $\mathbf{B}^{\dagger}=\underset{r>0}{\cup}\mathbf{B}^{\dagger,r}$ and $\mathbf{B}_K^{\dagger}=\underset{r>0}{\cup}\mathbf{B}_K^{\dagger,r}$.

It can be shown that

$$\mathbf{B}_{K_0}^{\dagger,r}=\left\{f(\pi)=\sum_{k\in\mathbb{Z}}a_k\pi^k\;\mid\;a_k\in K_0\text{ and }f\text{ is holomorphic and bounded on }p^{-1/r}\leqslant|X|_p<1\right\}.$$

More generally, let F be the maximal unramified subextension of K_∞/K_0 and let $e=[K_\infty:K_0(\zeta_{p^\infty})]$. Then there exists $r(K)$ and $\pi_K\in\mathbf{B}_K^{\dagger,r(K)}$ such that for any $r\geqslant r(K)$ one has

$$\mathbf{B}_K^{\dagger,r}=\left\{f=\sum_{k\in\mathbb{Z}}a_k\pi_K^k\;\mid\;a_k\in F\text{ and }f\text{ is holomorphic and bounded on }p^{-1/er}\leqslant|X|_p<1\right\}.$$

Define

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}\;=\;\left\{f=\sum_{k\in\mathbb{Z}}a_k\pi_K^k\;\mid\;a_k\in F\text{ and }f\text{ is holomorphic on }p^{-1/er}\leqslant|X|_p<1\right\}\quad\text{if }r\geqslant r(K)$$

and set $\mathcal{R}(K)=\underset{r\geqslant r(K)}{\cup}\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$. It is not difficult to show that $\mathbf{B}_K^{\dagger}$ and $\mathcal{R}(K)$ are stable under the actions of Γ and φ.

1.1.2. **(φ,Γ)-modules** (see [F1], [Cz2], [Cz4]). Let A be a commutative ring equipped with a Frobenius φ. A finitely generated free A-module M is said to be a φ-module if it is equipped with a φ-semilinear map $\varphi\,:\, M \longrightarrow M$ such that the induced map $A\otimes_{\varphi} M \longrightarrow M$ is an isomorphism. If $r\in \mathbb{Q}$ is a rational written in the form $r=a/b$ such that $a\in \mathbb{Z}$, $b\in \mathbb{N}^*$ and $(a,b)=1$, we denote by $D^{[r]}$ the φ-module of rank b defined by

$$\varphi(e_1)=e_2,\;\varphi(e_2)=e_3,\dots,\varphi(e_{b-1})=e_b,\;\varphi(e_b)=p^ae_1.$$

A φ module is said to be elementary, if it is isomorphic to $D^{[r]}$ for some r.

Let $\mathbf{k}$ be an algebraically closed field and let $\mathcal{K}=W(\mathbf{k})\left[1/p\right]$. The theory of φ-modules over $\mathcal{K}$ goes back to Dieudonné and Manin [Mn]. Namely, for any φ-module D over $\mathcal{K}$ there exists a unique decomposition into a direct sum of elementary modules

$$D\simeq \underset{i\in I}{\oplus} D^{[r_i]}.$$

The rationals r_i are called slopes of D. One says that D is pure of slope r if all $r_i=r$. In particular, if D is a φ-module over $\mathbf{B}^{\dagger}_K$, then it can be decomposed over $\widetilde{\mathbf{B}}$:

$$D\otimes_{\mathbf{B}^{\dagger}_K}\widetilde{\mathbf{B}}\,\simeq\,\underset{i\in I}{\oplus} D^{[r_i]}.$$

We say that a φ-module over $\mathbf{B}^{\dagger}_K$ is etale if it is pure of slope 0.

The analogue of this theory over the Robba ring $\mathcal{R}(K)$ is higher non trivial. It was found by Kedlaya [Ke]. Let D be a φ-module over $\mathcal{R}(K)$. Then there exists a canonical filtration

$$D_0\subset D_1\subset \cdots \subset D_h=D$$

and for any $1\leqslant i\leqslant h$, a unique $\mathbf{B}^{\dagger}_K$-submodule $\Delta_i(D)$ of D_i/D_{i-1} satisfying the following properties:

i) $\Delta_i(D)$ is a pure φ-module of slope r_i such that $D_i/D_{i-1}=\Delta_i(D)\otimes_{\mathbf{B}^{\dagger}_K}\mathcal{R}(K)$.

ii) One has

$$r_1<r_2<\cdots<r_h.$$

We say that D is pure of slope r if $h=1$ and $r_1=r$. In this case there exists a unique pure $\mathbf{B}^{\dagger}_K$-module $\Delta(D)$ of slope r such that $D=\Delta(D)\otimes_{\mathbf{B}^{\dagger}_K}\mathcal{R}(K)$.

Now assume that A is a commutative ring endowed with actions of φ and Γ commuting to each other. A (φ,Γ)-module over A is a φ-module equipped with a semilinear action of Γ commuting with φ.

Proposition 1.1.3. *The functors $\Delta\mapsto \Delta\otimes_{\mathbf{B}^{\dagger}_K}\mathcal{R}(K)$ and $D\mapsto \Delta(D)$ are quasi-inverse equivalences between the category of etale (φ,Γ)-modules over $\mathbf{B}^{\dagger}_K$ and the category of (φ,Γ)-modules over $\mathcal{R}(K)$ of slope 0.*

Proof. see [Cz4], Proposition 1.4 and Corollary 1.5.

Proposition 1.1.4. *i) The functor*

$$\mathbf{D}^{\dagger}\;:\;V\mapsto \mathbf{D}^{\dagger}(V)=(\mathbf{B}^{\dagger}\otimes_{\mathbb{Q}_p}V)^{H_K}$$

establishes an equivalence between the category of p-adic representations of G_K and the category of etale (φ,Γ)-modules over $\mathbf{B}^{\dagger}_K$.

ii) The functor $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)=\mathcal{R}(K)\otimes_{\mathbf{B}^{\dagger}_K}\mathbf{D}^{\dagger}(V)$ establishes an equivalence between the category of p-adic representations of G_K and the category of (φ,Γ)-modules over $\mathcal{R}(K)$ of slope 0.

Proof. The first statement is Fontaine's classification of p-adic representations [F1] together with the main theorem of [CC]. The second statement follows from i) together with proposition 1.1.3. See [Cz4], Proposition 1.7 for details.

1.1.5. Cohomology of (φ,Γ)-modules (see [H1], [H2], [Li]). Fix a generator γ of Γ. If D is a (φ,Γ)-module over a ring A, we denote by $C_{\varphi,\gamma}(D)$ the complex

$$C_{\varphi,\gamma}(D)\ :\ 0\rightarrow D\xrightarrow{f} D\oplus D\xrightarrow{g} D\rightarrow 0$$

where $f(x)=((\varphi-1)\,x,(\gamma-1)\,x)$ and $g(y,z)=(\gamma-1)\,y-(\varphi-1)\,z$. We shall write $H^i(D)$ for the cohomology of $C_{\varphi,\gamma}(D)$. A short exact sequence of (φ,Γ)-modules

$$0\rightarrow D'\rightarrow D\rightarrow D''\rightarrow 0$$

gives rise to a long exact sequence of cohomology groups:

$$0\rightarrow H^0(D')\rightarrow H^0(D)\rightarrow H^0(D'')\rightarrow H^1(D')\rightarrow\cdots\rightarrow H^2(D'')\rightarrow 0.$$

If D_1 and D_2 are (φ,Γ)-modules define a bilinear map

$$\cup\ :\ H^i(D_1)\times H^j(D_2)\rightarrow H^{i+j}(D_1\otimes D_2)$$

by

$$\begin{aligned}
&\mathrm{cl}(x_1)\cup\mathrm{cl}(x_2)=\mathrm{cl}(x_1\otimes x_2)\qquad \text{if } i=j=0,\\
&\mathrm{cl}(x_1)\cup\mathrm{cl}(x_2,y_2)=\mathrm{cl}(x_1\otimes x_2,x_1\otimes y_2)\qquad \text{if } i=0,\ j=1,\\
&\mathrm{cl}(x_1,y_1)\cup\mathrm{cl}(x_2,y_2)=\mathrm{cl}(y_1\otimes\gamma(x_2)-x_1\otimes\varphi(y_2))\qquad \text{if } i=1,\ j=1.
\end{aligned}$$

For any (φ,Γ)-module D let $D(\chi)$ denote the φ-module D endowed with the action of Γ on D twisted by the cyclotomic character χ. Set $D^*=\mathrm{Hom}_{\mathcal{R}(K)}(D,\mathcal{R}(K))$.

The following theorem extends the main results of [H1], [H2] to (φ,Γ)-modules over $\mathcal{R}(K)$.

Theorem 1.1.6. *If D be a (φ,Γ)-module over $\mathcal{R}(K)$ then*
i) $H^i(D)$ are finite dimensional $\mathbb{Q}_p$-vector spaces and

$$\chi(D)=\sum_{i=0}^{2}(-1)^i\dim_{\mathbb{Q}_p}H^i(D)\ =\ -[K:\mathbb{Q}_p]\,\mathrm{rg}(D)$$

ii) For $i=0,1,2$ the cup product

$$H^i(D)\times H^{2-i}(D^*(D(\chi)))\xrightarrow{\cup} H^2(\mathcal{R}(K)(\chi))$$

is a perfect pairing into $H^2(\mathcal{R}(K)(\chi))\simeq\mathbb{Q}_p$.
iii) If D is an etale (φ,Γ)-module over $\mathbf{B}_K^{\dagger}$ then the natural map

$$C_{\varphi,\gamma}(D)\rightarrow C_{\varphi,\gamma}(D\otimes_{\mathbf{B}_K^{\dagger}}\mathcal{R}(K))$$

is a quasi isomorphism. In particular, if V is p-adic representation of G_K, the Galois cohomology $H^*(K,V)$ is canonically and functorially isomorphic to $H^*(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))$.

Proof. See [Li], Theorems 1.1 and 1.2.

Remark 1.1.7. The isomorphism $H^2(\mathcal{R}(K)(\chi))\simeq \mathbb{Q}_p$ is constructed in [H2]. If K is unramified over $\mathbb{Q}_p$ it is given by the formula

$$\mathrm{cl}(\alpha)\mapsto -\left(1-\frac{1}{p}\right)^{-1}(\log\chi(\gamma))^{-1}\,\mathrm{res}\frac{\alpha d\pi}{1+\pi}.$$

It can be shown that this definition is compatible with the canonical isomorphism $H^2(\mathbb{Q}_p,\mathbb{Q}_p(1))\simeq\mathbb{Q}_p$ of the local class field theory (see [Ben1], section 2.2).

1.2. Semistable and crystalline (φ,Γ)**-modules** (see [Ber1], [Ber2], [BC]).

1.2.1. Let $\log\pi$ be a transcendental element over the field of fractions of $\mathcal{R}(K)$ equipped with the following actions of φ and Γ

$$\varphi(\log\pi)=p\log\pi+\log\left(\frac{\varphi(\pi)}{\pi^p}\right),$$
$$\gamma(\log\pi)=\log\pi+\log\left(\frac{\gamma(\pi)}{\pi}\right).$$

Remark that the series $\log\left(\varphi(\pi)/\pi^p\right)$ and $\log\left(\gamma(\pi)/\pi\right)$ converge in $\mathcal{R}(K)$. Set $\mathcal{R}_{\log}(K)=\mathcal{R}(K)[\log\pi]$ and define a monodromy operator $N\,:\,\mathcal{R}_{\log}(K)\to\mathcal{R}_{\log}(K)$ by $N=-\left(1-\dfrac{1}{p}\right)^{-1}\dfrac{d}{d\log\pi}$. Write $K_\infty((t))$ for the ring of Laurent power series equipped with the filtration $\mathrm{Fil}^iK_\infty((t))=t^iK_\infty[[t]]$ and the natural action of Γ given by $\gamma\left(\sum a_it^i\right)=\sum\gamma(a_i)\chi(\gamma)^it^i$. Recall that for any $n\geqslant 0$ and $r_n=p^{n-1}(p-1)$ there exists a well defined injective homomorphism

$$\iota_n=\varphi^{-n}\ :\ \mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n}\to K_\infty[[t]]$$

which is characterized by the fact that $\iota_n(\pi)=\zeta_{p^n}e^{t/p^n}-1$ (see for example [Ber1], §2.4). For any $r>0$ let $n(r)$ denote the smallest integer n such that $r_n\geqslant r$.

Lemma 1.2.2. *Let D be a (φ,Γ)-module over $\mathcal{R}(K)$. There exists $r(D)>0$ such that for any $r\geqslant r(D)$ there exists a unique free $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$-submodule $D^{(r)}$ of D stable under Γ and having the following properties:*

i) If $s>r$ then $D^{(s)}\simeq D^{(r)}\otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}}\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$.

ii) $D^{(r)}\otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}}\mathcal{R}(K)\simeq D$;

iii) $D^{(rp)}\simeq \mathbf{B}_{\mathrm{rig},K}^{\dagger,rp}\otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r},\varphi}D^{(r)}$.

Proof. This is Theorem 1.3.3 of [Ber2].

1.2.3. Let D be a (φ,Γ)-module over $\mathcal{R}(K)$. Following [BC], define

$$\mathcal{D}_{\mathrm{dR}}(D)\,=\,(K_\infty((t))\otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r},\,\iota_n}D^{(r)})^\Gamma,\qquad r\geqslant r(D),\ n\geqslant n(r).$$

From Lemma 1.2.2 it follows that this definition does not depend on the choice of r and n. Since $K_\infty((t))^\Gamma=K$, a standard argument shows that $\mathcal{D}_{\mathrm{dR}}(D)$ is a K-vector space such that

$$\dim_K\mathcal{D}_{\mathrm{dR}}(D)\leqslant\mathrm{rg}(D).$$

Moreover, $\mathcal{D}_{\mathrm{dR}}(D)$ is equipped with the induced filtration

$$\mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(D)\,=\,(t^iK_\infty[[t]]\otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r},\,\iota_n}D^{(r)})^\Gamma,\qquad n\geqslant n(r).$$

The jumps of this filtration (with multiplicities) will be called Hodge-Tate weights of D . Next, we define

$$\mathcal{D}_{\mathrm{cris}}(D) = \left(D \otimes_{\mathcal{R}(K)} \mathcal{R}(K)[1/t]\right)^{\Gamma} = (D[1/t])^{\Gamma},$$
$$\mathcal{D}_{\mathrm{st}}(D) = \left(D \otimes_{\mathcal{R}(K)} \mathcal{R}_{\log}(K)[1/t]\right)^{\Gamma}.$$

Then $\mathcal{D}_{\mathrm{cris}}(D)$ is a finite dimensional K_0-vector space equipped with a linear action of φ. For any $r \geqslant r(D)$ and $n \geqslant n(r)$ the map ι_n induces a natural inclusion $\mathcal{D}_{\mathrm{cris}}(D) \hookrightarrow \mathcal{D}_{\mathrm{dR}}(D)$ and we set

$$\mathrm{Fil}^i \mathcal{D}_{\mathrm{cris}}(D) = \varphi^n(\iota_n(\mathcal{D}_{\mathrm{cris}}(D)) \cap \mathrm{Fil}^i \mathcal{D}_{\mathrm{dR}}(D)).$$

It is easy to see that $\mathrm{Fil}^i\mathcal{D}_{\mathrm{cris}}(D)$ is a filtration on $\mathcal{D}_{\mathrm{cris}}(D)$ which does not depend on the choice of n and r. The same arguments show that $\mathcal{D}_{\mathrm{st}}(D)$ is a finite dimensional K_0-vector space equipped with natural actions of φ and N and a filtration $\mathrm{Fil}^i\mathcal{D}_{\mathrm{st}}(D)$ induced from $\mathcal{D}_{\mathrm{dR}}(D)$. From definitions it follows immediately that $\mathcal{D}_{\mathrm{cris}}(D) = \mathcal{D}_{\mathrm{st}}(D)^{N=0}$. Moreover

$$\dim_{K_0} \mathcal{D}_{\mathrm{cris}}(D) \leqslant \dim_{K_0} \mathcal{D}_{\mathrm{st}}(D) \leqslant \dim_K \mathcal{D}_{\mathrm{dR}}(D) \leqslant \mathrm{rg}(D).$$

Definition. *We say that D is crystalline (resp. semistable, resp. de Rham) if* $\dim_{K_0} \mathcal{D}_{\mathrm{cris}}(D) = \mathrm{rg}_{\mathcal{R}(K)}(D)$ *(resp.* $\dim_{K_0} \mathcal{D}_{\mathrm{st}}(D) = \mathrm{rg}_{\mathcal{R}(K)}(D)$*, resp.* $\dim_K \mathcal{D}_{\mathrm{dR}}(D) = \mathrm{rg}_{\mathcal{R}(K)}(D)$*).*

This definition is motivated by the following proposition which summarizes the results of Berger about the classification of p-adic representations in terms of (φ, Γ)-modules.

Proposition 1.2.4. *Let V be a p-adic representation of G_K. Then*

$$\mathbf{D}_*(V) \simeq \mathcal{D}_*(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)), \qquad \text{where } * \in \{\mathrm{dR}, \mathrm{st}, \mathrm{cris}\}.$$

In particular, V is a de Rham (resp. crystalline, resp. semistable) if and only if $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ is.

Proof. see [Ber1], Theorem 0.2 and Proposition 5.9.

1.2.5. Let L/K be a finite extension and $\Gamma_L = \mathrm{Gal}(L_\infty/L)$. Remark that $\mathcal{R}(K) \subset \mathcal{R}(L)$. If D is a (φ,Γ)-module over $\mathcal{R}(K)$ we set $D_L = \mathcal{R}(L) \otimes_{\mathcal{R}(K)} D$. It is easy to see that D_L has a natural structure of (φ, Γ_L)-module. We say that D is potentially semistable if there exists L/K such that D_L is semistable. For any D define

$$\mathcal{D}_{\mathrm{pst}}(D) = \varinjlim_{L/K} \mathcal{D}_{\mathrm{st}/L}(D_L).$$

Then $\mathcal{D}_{\mathrm{pst}}(D)$ is a finite dimensional K_0^{nr}-vector space equipped with natural actions of φ, N and a discrete action of G_K. It is easy to see that D is potentially semistable if and only if $\dim_{K_0^{nr}} \mathcal{D}_{\mathrm{pst}}(D) = \mathrm{rg}(D)$.

Proposition 1.2.6. *Let D be a (φ, Γ)-module. The following statements are equivalent:*

1) D is potentially semistable.

2) D is a de Rham.

Proof. If D is etale, this is the p-adic monodromy conjecture stated by Fontaine. In [Ber1], Corollary 5.22 Berger deduced it from Crew's conjecture which was proved independently by André, Mebkhout et Kedlaya. Remark that Berger's arguments work if D is not more etale and give therefore a proof of the Proposition. See especially Lemma 5.13 and Propositions 5.14 and 5.15 of [Ber1].

1.2.7. A filtered (φ, N)-module over K is a finite dimensional K_0-vector space M equipped with the following structures:

- an exhaustive decreasing filtration $(\mathrm{Fil}^i M_K)$ on $M_K = K \otimes_{K_0} M$;
- a σ-semilinear bijective map $\varphi\,:\, M \to M$;
- a K-linear nilpotent operator $N\,:\, M \to M$ such that $N\varphi = p\varphi N$.

A filtered (φ, N, G_K)-module over K is a (φ, N)-module M over K^{nr} equipped with a semilinear action of G_K which is discrete on the inertia subgroup $I_K = \mathrm{Gal}(\overline{K}/K^{nr})$ and an exhaustive decreasing filtration $(\mathrm{Fil}^i M_{K^{nr}})$ on $M_{K^{nr}} = K^{nr} \otimes_{K_0^{nr}} M$ stable under the action of G_K. In particular the filtration on $M_{K^{nr}}$ is completely defined by the filtration $\mathrm{Fil}^i M_K = M_K \cap \mathrm{Fil}^i M_{K^{nr}}$ on the K-vector space $M_K = (M_{K^{nr}})^{G_K}$.

A K-linear map $f\,:\, M \to M'$ is said to be a morphism of filtered (φ, N, G_K)-modules if f commutes with φ, N and G_K and $f(\mathrm{Fil}^i M_K) \subset \mathrm{Fil}^i M'_K$ for all $i \in \mathbb{Z}$.

The category $\mathbf{MF}_K^{\varphi,N,G_K}$ of filtered (φ, N)-modules is additive, has kernels and cokernels but is not abelian. Moreover it is equipped with the tensor product defined by

$$\begin{aligned}
&\mathrm{Fil}^i(M \otimes M')_K = \sum_{j+k=i} \mathrm{Fil}^j M_K \otimes_{K^{nr}} \mathrm{Fil}^k M'_K,\\
&\varphi(m \otimes m') = \varphi(m) \otimes \varphi(m'),\\
&N(m \otimes m') = N(m) \otimes m' + m \otimes N(m').
\end{aligned}$$

Denote by $\mathbf{1}$ the vector space K_0^{nr} equipped with the natural actions of φ and G_K , the trivial action of N and the filtration given by

$$\mathrm{Fil}^i \mathbf{1}_K = \begin{cases} K & \text{if } i \leqslant 0,\\ 0 & \text{if } i > 0. \end{cases}$$

Then $\mathbf{1}$ is a unit object of $\mathbf{MF}_K^{\varphi,N,G_K}$, i.e.

$$\mathbf{1} \otimes M \simeq M \otimes \mathbf{1} \simeq M$$

for any M. The functor $M \mapsto K_0^{nr} \otimes_{K_0} M$ is a full embedding of $\mathbf{MF}_K^{\varphi,N}$ into $\mathbf{MF}_K^{\varphi,N,G_K}$.

A filtered Dieudonné module is an object $M \in \mathbf{MF}_K^{\varphi,N}$ such that the operator N is trivial on M. Filtered Dieudonné modules form a full subcategory $\mathbf{MF}_K^{\varphi}$ of $\mathbf{MF}_K^{\varphi,N}$. Denote by $\mathbf{M}_{\mathrm{cris},K}^{\varphi,\Gamma}$, $\mathbf{M}_{\mathrm{st},K}^{\varphi,\Gamma}$ and $\mathbf{M}_{\mathrm{pst},K}^{\varphi,\Gamma}$ the categories of crystalline, semistable and potentially semistable (φ,Γ)-modules respectively. For all these categories we will use the following convention. A sequence of objects

$$0 \to A_1 \to A \to A_2 \to 0$$

is said to be exact if A/A_1 is isomorphic to A_2.

Lemma 1.2.8. *Let*

$$0 \to D_1 \to D \to D_2 \to 0 \tag{2}$$

be an exact sequence of (φ,Γ)*-modules. If* D *is a de Rham (resp. crystalline, resp. semistable) , then* D_1 *and* D_2 *are de Rham (resp. crystalline, resp. semistable) and the sequence*

$$0 \to \mathcal{D}_*(D_1) \to \mathcal{D}_*(D) \to \mathcal{D}_*(D_2) \to 0, \qquad * \in \{\mathrm{dR}, \mathrm{cris}, \mathrm{st}\}$$

is exact in the category of filtered K*-vector spaces (resp. in* $\mathbf{MF}_K^{\varphi}$*, resp. in* $\mathbf{MF}_K^{\varphi,N}$*).*

Proof. The proof is standard. Assume that D is a de Rham module. By Lemma 2.2.11 of [BC], for r big enough the sequence

$$0 \to D_1^{(r)} \xrightarrow{\alpha} D^{(r)} \xrightarrow{\beta} D_2^{(r)} \to 0$$

is exact. Tensoring this sequence with $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ and taking invariants, one obtains an exact sequence of K-vector spaces

$$0 \longrightarrow \mathcal{D}_{\mathrm{dR}}(D_1) \xrightarrow{\alpha} \mathcal{D}_{\mathrm{dR}}(D) \xrightarrow{\beta} \mathcal{D}_{\mathrm{dR}}(D_2).$$

As $\dim_K \mathcal{D}_{\mathrm{dR}}(D) = \mathrm{rg}(D)$ and $\dim_K \mathcal{D}_{\mathrm{dR}}(D_i) \leqslant \mathrm{rg}(D_i)$ $(i = 1,2)$, this implies that $\dim_K \mathcal{D}_{\mathrm{dR}}(D_i) = \mathrm{rg}(D_i)$. Thus D_i are de Rham modules. Next, the standard argument involving Artin's trick shows that for any de Rham (φ,Γ)-module D one has an isomorphism

$$D^{(r)} \otimes_{\mathcal{R}(K),\,\iota_n} K_\infty((t)) \simeq \mathcal{D}_{\mathrm{dR}}(D) \otimes_K K_\infty((t)).$$

Thus, for any $k \in \mathbb{Z}$

$$D^{(r)} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\,\iota_n} t^k K_\infty[[t]] \,=\, \sum_{i\in\mathbb{Z}} \mathrm{Fil}^{-i}\mathcal{D}_{\mathrm{dR}}(D) \otimes_K t^{k+i} K_\infty[[t]].$$

Since

$$H^1(\Gamma, K_\infty t^m) \,=\, \begin{cases} 0 & \text{if } m \neq 0 \\ K & \text{if } m = 0 \end{cases}$$

(see [T], Proposition 8), we obtain that

$$H^1\left(\Gamma, D^{(r)} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\,\iota_n} t^k K_\infty[[t]]\right) \,=\, \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D).$$

The short exact sequence (2) induces a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D_1) \xrightarrow{\alpha} \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D) \xrightarrow{\beta} \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D_2) \xrightarrow{\delta} \\ H^1\left(\Gamma, D_1^{(r)} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\,\iota_n} t^k K_\infty[[t]]\right) \xrightarrow{\alpha_*} H^1\left(\Gamma, D^{(r)} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\,\iota_n} t^k K_\infty[[t]]\right) \longrightarrow \cdots . \end{aligned}$$

Since the diagram

$$\begin{array}{ccc} H^1\left(\Gamma, D_1^{(r)} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\,\iota_n} t^k K_\infty[[t]]\right) & \xrightarrow{\alpha_*} & H^1\left(\Gamma, D^{(r)} \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\,\iota_n} t^k K_\infty[[t]]\right) \\ {\scriptstyle\simeq}\downarrow & & {\scriptstyle\simeq}\downarrow \\ \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D_1) & \xrightarrow{\mathrm{id}} & \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D) \end{array}$$

is commutative, the map α_* is injective and we obtain that for all k the sequence

$$0 \longrightarrow \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D_1) \xrightarrow{\alpha} \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D) \xrightarrow{\beta} \mathrm{Fil}^k\mathcal{D}_{\mathrm{dR}}(D_2) \longrightarrow 0$$

is exact. Thus, the sequence $0 \to \mathcal{D}_{\mathrm{dR}}(D_1) \to \mathcal{D}_{\mathrm{dR}}(D) \to \mathcal{D}_{\mathrm{dR}}(D_2) \to 0$ is exact in the category of filtered K-vector spaces. The case of crystalline (resp. semistable) modules is analogous and is omitted here.

Proposition 1.2.9. *The functors*

$$\begin{cases} \mathcal{D}_{\mathrm{pst}} \;:\; \mathbf{M}^{\varphi,\Gamma}_{\mathrm{pst},K} \longrightarrow \mathbf{MF}^{\varphi,N,G_K}_K, \\ D \mapsto \mathcal{D}_{\mathrm{pst}}(D), \end{cases}$$

$$\begin{cases} \mathcal{D}_{\mathrm{st}} \ : \ \mathbf{M}_{\mathrm{st},K}^{\varphi,\Gamma} \to \mathbf{MF}_K^{\varphi,N}, \\ D \mapsto \mathcal{D}_{\mathrm{st}}(D) \end{cases}$$

and

$$\begin{cases} \mathcal{D}_{\mathrm{cris}} \ : \ \mathbf{M}_{\mathrm{cris},K}^{\varphi,\Gamma} \to \mathbf{MF}_K^{\varphi}, \\ D \mapsto \mathcal{D}_{\mathrm{cris}}(D) \end{cases}$$

are equivalences of categories.

Proof. This proposition is a reformulation of the main result of [Ber2]. Let ∇ denote the operator $\dfrac{\log(\gamma)}{\log\chi(\gamma)}$ and let $\partial = t^{-1}\nabla$. Remark that ∂ acts as $(1+\pi)\,\dfrac{d}{d\pi}$ on $\mathcal{R}(K)$. In op. cit. Berger establishes an equivalence $\mathcal{S}$ between the category of (φ,Γ)-modules over $\mathcal{R}(K)$ such that $\mathrm{Lie}(\Gamma)$ acts locally trivially and the category of filtered (φ,N,G_K)-modules. Namely, if D is such a (φ,Γ)-module, then there exists a unique submodule $A\subset D$ such that $\partial(A)\subset A$ and $A[1/t]=D[1/t]$ and $\mathcal{S}(D)$ is defined as the space of L-solutions of the differential equation (A,∂) for L/K big enough.

Consider the natural injection

$$\mathcal{D}_{\mathrm{st}/L}(D)\otimes_{L_0}\mathcal{R}_{\log}(L)[1/t]\hookrightarrow D_L\otimes_{\mathcal{R}(L)}\mathcal{R}_{\log}(L)[1/t]$$

If D is semistable over L set $B=(\mathcal{D}_{\mathrm{pst}/L}(D)\otimes_{L_0}\mathcal{R}_{\log}(L))^{N=0}$. Then B is a (φ,Γ_L)-module over $\mathcal{R}(L)$ of rank $\mathrm{rg}_{\mathcal{R}(K)}(D)$ which is stable under ∂. Let $A=B^{H_K}$. By Hilbert theorem 90 we have $B=\mathcal{R}(L)\otimes_{\mathcal{R}(K)}A$. Then $\partial(A)\subset A$, $A[1/t]=D[1/t]$ and the space of L-solutions of the differential equation (A,∂) is

$$(A\otimes_{\mathcal{R}(K)}\mathcal{R}_{\log}(L))^{\partial=0}=\mathcal{D}_{\mathrm{st}/L}(D).$$

Thus, $\mathcal{S}(D)=\mathcal{D}_{\mathrm{st}/L}(D)$.

Conversely, let M be a filtered (φ,N,G_K)-module and let L/K be a finite extension such that I_L acts trivially on M. Then $M'=M^{G_L}$ is an L_0-lattice of M and we set

$$\mathbf{D}=(M'\otimes_{L_0}\mathcal{R}_{\log}(L))^{N=0}.$$

It is easy to see that $\mathbf{D}$ is a (φ,Γ_L)-module of rank $\dim_{K_0^{nr}}(M)$. By Theorem II.1.2 of [Ber 2] there exists a unique (φ,Γ_L)-module $\mathcal{M}_L(M)\subset\mathbf{D}$ such that $\mathcal{M}_L(M)[1/t]=\mathbf{D}[1/t]$ and

$$L_n[[t]]\otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},L},\iota_n}\mathcal{M}_L(M)^{(r)}=\mathrm{Fil}^0(L_n((t))\otimes_{L_0,\varphi^{-n}}M')\qquad\text{for all } n\geqslant n(r).$$

Set $\mathcal{M}(M)=\mathcal{M}_L(M)^{H_K}$. Berger proves that the functor $M\mapsto\mathcal{M}(M)$ is a quasi-inverse to $\mathcal{S}$. From the isomorphism $\mathbf{D}\otimes_{\mathcal{R}(L)}\mathcal{R}_{\log}(L)\simeq M\otimes_{L_0}\mathcal{R}_{\log}(L)$ it follows easily that

$$\mathcal{D}_{\mathrm{st}/L}(\mathcal{M}(M)_L)=(\mathbf{D}\otimes_{\mathcal{R}_L}\mathcal{R}_{\log}(L))^{\Gamma_L}=M'.$$

Thus D is a semistable (φ,Γ)-module such that $\mathcal{D}_{\mathrm{pst}}(D)=M$ and we proved that $\mathcal{D}_{\mathrm{pst}}$ is an equivalence between the category of potentially semistable (φ,Γ)-modules and $\mathbf{MF}_K^{\varphi,N,G_K}$. Passing to the subcategory $\mathbf{M}_{\mathrm{st},K}^{\varphi,\Gamma}$ (resp. $\mathbf{M}_{\mathrm{cris},K}^{\varphi,\Gamma}$) we obtain immediately that $\mathcal{D}_{\mathrm{st}}$ (resp. $\mathcal{D}_{\mathrm{cris}}$) is an equivalence of the category of semistable (resp. crystalline) modules onto $\mathbf{MF}_K^{\varphi,N}$ (resp. $\mathbf{MF}_K^{\varphi}$). The proposition is proved.

1.3. Triangulation of (φ,Γ)**-modules** (see [Ber2], [Cz4], [BC]).

1.3.1. The results of this section will not be used in the remainder of this paper. Nevertheless the notion of a trianguline representation is closely related to our definition of the $\mathcal{L}$-invariant and we review it here. For simplicity we assume that $K=\mathbb{Q}_p$ and write $\mathcal{R}$ for $\mathcal{R}(\mathbb{Q}_p)$ but fix a finite extension L of $\mathbb{Q}_p$ as the coefficient field. Equip L with trivial actions of φ and Γ and set $\mathcal{R}_L=\mathcal{R}_{\mathbb{Q}_p}\otimes L$. Remark that the theory of sections 1.1 and 1.2 extends without difficulty to (φ,Γ)-modules over $\mathcal{R}_L$. Let D be such a module. A triangulation of D is a strictly increasing filtration

$$\{0\}=F_0D\subset\cdots\subset F_iD\subset F_{i+1}D\subset\cdots\subset F_dD=D$$

by (φ,Γ)-submodules over $\mathcal{R}_L$ such that

- every F_iD is a saturated submodule of D;
- the factor modules $\mathrm{gr}_i(D)=F_iD/F_{i-1}D$ are free of rank 1.

Triangular modules were first studied in [Cz4].

Now assume that D is semistable and that all the eigenvalues of $\varphi\,:\,\mathcal{D}_{\mathrm{st}}(D)\to\mathcal{D}_{\mathrm{st}}(D)$ are in L. Following Mazur [Mr2], a refinement of D is a filtration on $\mathcal{D}_{\mathrm{st}}(D)$

$$\{0\}=\mathcal{F}_0\mathcal{D}_{\mathrm{st}}(D)\subset\mathcal{F}_1\mathcal{D}_{\mathrm{st}}(D)\subset\cdots\subset\mathcal{F}_d\mathcal{D}_{\mathrm{st}}(D)=\mathcal{D}_{\mathrm{st}}(D)$$

by L-subspaces stable under φ and N and such that each factor $\mathrm{gr}_i\mathcal{D}_{\mathrm{st}}(D)$ is of dimension 1. Any refinement fixes an ordering $\alpha_1,\dots,\alpha_d$ of eigenvalues of φ and an ordering $k_1,\dots,k_d$ of Hodge weights of D taken with multiplicities.

Proposition 1.3.2. *Let D be a semistable (φ,Γ)-module over $\mathcal{R}_L$.*

i) The equivalence between the category of semistable modules and the category of filtered (φ,N)-modules induces a bijection between the set of triangulations of D and the set of refinements of D.

ii) If $(F_iD)_{i\in\mathbb{Z}}$ is the triangulation associated to a refinement $(\mathcal{F}_i\mathcal{D}_{\mathrm{st}}(D))_{i\in\mathbb{Z}}$ then for each i the factor gr_iD is isomorphic to $\mathcal{R}_L(\delta_i)$ where δ_i is defined by $\delta_i(p)=\alpha_ip^{-k_i}$ and $\delta_i(u)=u^{-k_i}$ $(u\in\mathbb{Z}_p^*)$.

Proof. For crystalline representations this Proposition was proved in [BC], Proposition 2.4.1 and the same proof works in the general case. On the other hand, it can be deduced easily from Proposition 1.2.9. Indeed, the first statement is obvious. Next, let e_i denote the canonical generator of $\mathcal{R}_L(\delta_i)$. Then $\mathcal{D}_{\mathrm{st}}(\mathcal{R}_L(\delta_i))$ is the one-dimensional L vector space generated by $m_i=t^{k_i}\otimes e_i$. One has $\varphi(m_i)=\alpha_im_i$, $Nm_i=0$ and k_i is the unique Hodge number of $\mathcal{D}_{\mathrm{st}}(\mathcal{R}_L(\delta_i))$. Thus $\mathcal{D}_{\mathrm{st}}(\mathcal{R}_L(\delta_i))\simeq\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_i(D))$ and gr_iD is isomorphic to $\mathcal{R}_L(\delta_i)$ by Proposition 1.2.9.

1.4. Crystalline and semistable extensions.

1.4.1. Let D be a (φ,Γ)-module over $\mathcal{R}(K)$. As usually, $H^1(D)$ can be described in terms of extensions. Namely, to any cocycle $\alpha=(a,b)\in Z^1(C_{\varphi,\gamma}(D))$ we can associate the extension

$$0\to D\to D_\alpha\to\mathcal{R}(K)\to 0$$

defined by

$$D_\alpha=D\oplus\mathcal{R}(K)\,e,\qquad(\varphi-1)\,e=a,\quad(\gamma-1)\,e=b\,.$$

This construction gives rise to an isomorphism

$$H^1(D)\simeq\mathrm{Ext}^1_{\mathcal{R}}(\mathcal{R}(K),D).$$

Definition. *We say that* $\mathrm{cl}(\alpha) \in H^1(D)$ *is crystalline (resp. semistable) if*

$$\dim_{K_0}\mathcal{D}_{\mathrm{cris}}(D_\alpha) = \dim_{K_0}\mathcal{D}_{\mathrm{cris}}(D) + 1$$

(resp. if $\dim_{K_0}\mathcal{D}_{\mathrm{st}}(D_\alpha) = \dim_{K_0}\mathcal{D}_{\mathrm{st}}(D) + 1$*) and define*

$$\begin{aligned} H^1_f(D) &= \{\mathrm{cl}(\alpha) \in H^1(D) \mid \mathrm{cl}(\alpha) \text{ is crystalline}\}, \\ H^1_{st}(D) &= \{\mathrm{cl}(\alpha) \in H^1(D) \mid \mathrm{cl}(\alpha) \text{ is semistable}\}. \end{aligned}$$

It is easy to see that $H^1_f(D) \subset H^1_{st}(D)$ are $\mathbb{Q}_p$-subspaces of $H^1(D)$.

Proposition 1.4.2. *Let* V *be a* p*-adic representation of* G_K*. Following Bloch and Kato* [BK] *define*

$$\begin{aligned} H^1_f(K,V) &= \ker(H^1(K,V) \to H^1(K, V\otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})), \\ H^1_{st}(K,V) &= \ker(H^1(K,V) \to H^1(K, V\otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})), \end{aligned}$$

where $\mathbf{B}_{\mathrm{cris}}$ *and* $\mathbf{B}_{\mathrm{st}}$ *are the rings of crystalline and semistable periods (see* [F2]*). Then* $H^1_f(K,V) \simeq H^1_f(\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ *and* $H^1_{st}(K,V) \simeq H^1_{st}(\mathbf{D}^\dagger_{\mathrm{rig}}(V))$.

Proof. For any cocycle $x \in Z^1(G_K, V)$ denote by V_x the corresponding extension of $\mathbb{Q}_p$ by V. It is well known (and easy to check) that $\mathrm{cl}(x) \in H^1_f(K,V)$ if and only if $\dim \mathbf{D}_{\mathrm{cris}}(V_x) = \dim \mathbf{D}_{\mathrm{cris}}(V) + 1$. Now the isomorphism $H^1_f(K,V) \simeq H^1_f(\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ follows from the isomorphism

$$\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p(0), V) \simeq \mathrm{Ext}^1_{\mathcal{R}}(\mathcal{R}(K), \mathbf{D}^\dagger_{\mathrm{rig}}(V))$$

given by Proposition 1.1.6 and Proposition 1.2.4. In the semistable case, the proof is analogous and is omitted here.

Lemma 1.4.3. *Let* D *be a* (φ,Γ)*-module. Then* $\mathrm{cl}(a,b) \in H^1_f(D)$ *(resp.* $\mathrm{cl}(a,b) \in H^1_{st}(D)$*) if and only if the equation* $(\gamma - 1)\, x = b$ *has a solution in* $D[1/t]$ *(resp. in* $D \otimes_{\mathcal{R}(K)} \mathcal{R}_{\log}(K)[1/t]$*).*

Proof. An extension D_α is crystalline (resp. semistable) if and only if there exists $x \in D[1/t]$ (resp. $x \in D\otimes_{\mathcal{R}(K)} \mathcal{R}_{\log}(K)[1/t]$) such that $x + e \in \mathcal{D}_{\mathrm{cris}}(D_\alpha)$ (resp. $x+e \in \mathcal{D}_{\mathrm{st}}(D_\alpha)$). As $(\gamma-1)\, e = b$, this proves the lemma.

The following proposition is proved (in a slightly different form) in [FP], Proposition 3.3.7 and [N], sections 1.19-1.21. For the convenience of the reader we recall the proof because it will be used in the proof of Proposition 1.5.8 below.

Proposition 1.4.4. *Let* D *be a potentially semistable* (φ,Γ)*-module. Then*
i) $H^0(D)$ *and* $H^1_f(D)$ *are canonically isomorphic to the cohomology of the complex*

$$C^\bullet_{\mathrm{cris}}(D) \ : \ \mathcal{D}_{\mathrm{cris}}(D) \xrightarrow{f} t_D(K) \oplus \mathcal{D}_{\mathrm{cris}}(D),$$

where $t_D(K) = \mathcal{D}_{\mathrm{dR}}(D)/\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)$ *and* $f(x) = (x \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)}, (1-\varphi)\, x)$.
ii) $H^0(D)$ *and* $H^1_{st}(D)$ *are canonically isomorphic to* $H^i(C^\bullet_{\mathrm{st}}(D))$ *(*$i=0,1$*) where*

$$C^\bullet_{\mathrm{st}}(D) \ : \ \mathcal{D}_{\mathrm{st}}(D) \xrightarrow{g} t_D(K) \oplus \mathcal{D}_{\mathrm{st}}(D) \oplus \mathcal{D}_{\mathrm{st}}(D) \xrightarrow{h} \mathcal{D}_{\mathrm{st}}(D),$$

with $g(x) = (x \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)}, (\varphi-1)\, x, N(x))$ *and* $h(x,y,z) = N(y) - (p\varphi - 1)\, z$.

Proof. i) Fix $r \gg 0$, $n \geqslant n(r)$ and consider the inclusion

$$\iota_n \ : \ \mathcal{D}_{\mathrm{cris}}(D) = (D^{(r)}[1/t])^\Gamma \hookrightarrow (K_n((t)) \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig},K},\, \iota_n} D^{(r)})^\Gamma \simeq \mathcal{D}_{\mathrm{dR}}(D).$$

By Lemmas 5.1 and 5.4 of [Ber1], $\iota_n(x) \in \mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(V)$ implies that $x \in D^{(r)}$. Thus

$$H^0(C^\bullet_{\mathrm{cris}}(D)) = (\mathrm{Fil}^0\mathcal{D}_{\mathrm{cris}}(D))^{\varphi=1} = (D^{(r)})^{\varphi=1,\gamma=1} = H^0(D).$$

Next, let D_α be a crystalline extension of D. Then we have exact sequences

$$\begin{aligned} &0 \to \mathcal{D}_{\mathrm{cris}}(D) \to \mathcal{D}_{\mathrm{cris}}(D_\alpha) \to K_0 \to 0, \\ &0 \to \mathcal{D}_{\mathrm{pst}}(D) \to \mathcal{D}_{\mathrm{pst}}(D_\alpha) \to K_0^{nr} \to 0. \end{aligned}$$

Then $\mathcal{D}_{\mathrm{cris}}(D_\alpha) = \mathcal{D}_{\mathrm{cris}}(D) \oplus K_0 e$ where $b = (\varphi - 1)\, e \in \mathcal{D}_{\mathrm{cris}}(D)$. The exact sequence

$$0 \to \mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D) \to \mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D_\alpha) \to K \to 0$$

shows that there exists $a \in \mathcal{D}_{\mathrm{dR}}(D)$ such that $e + a \in \mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D_\alpha)$. It is clear that a is unique modulo $\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)$. If we replace e by $e' = e + x$, $x \in \mathcal{D}_{\mathrm{cris}}(D)$, then $b' = (\varphi - 1)\, e' = b + (\varphi - 1)\, x$ and $a' = a - x$. Thus the class of $(a \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)}, b)$ modulo $\mathrm{Im}(f)$ does not depend on the choice of d and gives a well defined element of $H^1(C^\bullet_{\mathrm{cris}}(D))$. Moreover the filtered (φ, N, G_K)-module $\mathcal{D}_{\mathrm{pst}}(D_\alpha) = \mathcal{D}_{\mathrm{pst}}(D) \oplus K_0^{nr} e$ is completely defined by the class of $(a, b) \in t_D(K) \oplus \mathcal{D}_{\mathrm{cris}}(D)$ modulo $\mathrm{Im}(f)$ and the fact that G_K acts trivially on e. Conversely, to any $(a \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)}, b) \in t_D(K) \oplus \mathcal{D}_{\mathrm{cris}}(D)$ we can associate the extension $M = \mathcal{D}_{\mathrm{pst}}(D) \oplus K_0^{nr} e$ of filtered (φ, N, G_K)-modules defined by

$$\begin{aligned} &\varphi(e) = e + b, \quad N(e) = 0, \\ &\mathrm{Fil}^i M_K = \begin{cases} \mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(D), & \text{if } i < 0, \\ \mathrm{Fil}^i\mathcal{D}_{\mathrm{dR}}(D) + K(e + a), & \text{if } i \geqslant 0. \end{cases} \end{aligned}$$

By Proposition 1.2.9 there exists a potentially semistable (φ, Γ) module D' such that $\mathcal{D}_{\mathrm{pst}}(D') = M$. Then $\mathcal{D}_{\mathrm{cris}}(D') = M^{G_K, N=0} = \mathcal{D}_{\mathrm{cris}}(D) \oplus K_0 e$. Thus D' is a crystalline extension of $\mathcal{R}(K)$ by D and i) is proved.

ii) In the semistable case the proof is analogous. First remark that

$$H^0(C^\bullet_{\mathrm{st}}(D)) = (\mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(D))^{N=0,\varphi=1} = H^0(C^\bullet_{\mathrm{cris}}(D)) = H^0(D).$$

Next, if D_α is a semistable extension then we can write $\mathcal{D}_{\mathrm{st}}(D_\alpha) = \mathcal{D}_{\mathrm{st}}(D) \oplus K_0 e$ where $y = (\varphi - 1)e \in \mathcal{D}_{\mathrm{st}}(D)$ and $z = N(e) \in \mathcal{D}_{\mathrm{st}}(D)$. As in i) there exists $x \in \mathcal{D}_{\mathrm{dR}}(D)$ such that $e + x \in \mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D)$ and it is easy to see that the class of $(x \pmod{\mathcal{D}_{\mathrm{dR}}(D)}, y, z)$ modulo $\mathrm{Im}(g)$ does not depend on the choice of e and is a well defined element of $H^1(C^\bullet_{\mathrm{st}}(D))$. Now ii) follows from Proposition 1.2.9.

Corollary 1.4.5. *Let D be a potentially semistable (φ, Γ)-module. Then*

$$\begin{aligned} \dim_{\mathbb{Q}_p} H^1_f(D) - \dim_{\mathbb{Q}_p} H^0(D) \ &= \ \dim_{\mathbb{Q}_p} t_D(K), \\ \dim_{\mathbb{Q}_p} H^1_{st}(D) - \dim_{\mathbb{Q}_p} H^1_f(D) \ &= \ \dim_{\mathbb{Q}_p} \mathcal{D}_{\mathrm{cris}}(D^*(\chi))^{\varphi=1}. \end{aligned}$$

Proof. The first formula is obvious. The second follows from the fact that the cokernel of h is dual to the kernel of $\mathcal{D}_{\mathrm{st}}(D^*(\chi)) \xrightarrow{(N, 1-\varphi)} \mathcal{D}_{\mathrm{st}}(D^*(\chi)) \oplus \mathcal{D}_{\mathrm{st}}(D^*(\chi))$ which is $\mathcal{D}_{\mathrm{cris}}(D^*(\chi))^{\varphi=1}$.

Corollary 1.4.6. *Let*

$$0 \to D_1 \to D \to D_2 \to 0$$

be an exact sequence of potentially semistable (φ, Γ)-modules. Assume that one of the following conditions holds:

a) D is crystalline;

b) $\mathrm{Im}(H^0(D_2) \to H^1(D_1)) \subset H^1_f(D_1)$.

Then one has an exact sequence

$$0 \to H^0(D_1) \to H^0(D) \to H^0(D_2) \to H^1_f(D_1) \to H^1_f(D) \to H^1_f(D_2) \to 0. \tag{3}$$

Proof. i) If D is crystalline, then by Lemma 1.2.8 D_1 and D_2 are crystalline and we have an exact sequence of complexes

$$0 \to C^\bullet_{\mathrm{cris}}(D_1) \to C^\bullet_{\mathrm{cris}}(D) \to C^\bullet_{\mathrm{cris}}(D_2) \to 0.$$

Passing to cohomology we obtain (3). If the image of the connecting map is contained in $H^1_f(D_1)$ the sequence (3) is again well defined. Only the exactness at $H^1_f(D_2)$ requires proof, but it follows from the dimension argument using Corollary 1.4.5.

Lemma 1.4.7. *i) $D \mapsto H^i(C^\bullet_{\mathrm{st}}(D))$ defines a cohomological functor from $\mathbf{M}^{\varphi,\Gamma}_{pst,K}$ to the category of $\mathbb{Q}_p$-vector spaces. More precisely, let $0 \to D_1 \to D \to D_2 \to 0$ be an exact sequence of (φ, Γ)-modules. If D is potentially semistable, then D_1 and D_2 are potentially semistable and the sequence*

$$\begin{aligned} 0 \to H^0(D_1) \to H^0(D) \to H^0(D_2) \to H^1(C^\bullet_{st}(D_1)) \to H^1(C^\bullet_{st}(D)) \to H^1(C^\bullet_{st}(D_2)) \\ \to H^2(C^\bullet_{st}(D_1)) \to H^2(C^\bullet_{st}(D)) \to H^2(C^\bullet_{st}(D_2)) \to 0 \end{aligned}$$

is exact.

ii) The functor $D \mapsto H^i(C^\bullet_{\mathrm{st}}(D))$ is effacable. Namely for any $\mathrm{cl}(x) \in H^i(C^\bullet_{\mathrm{st}}(D))$ *($i = 1, 2$) there exists an exact sequence of semistable (φ, Γ)-modules $0 \to D \to D' \to D'' \to 0$ such that the image of* $\mathrm{cl}(x)$ *in $H^i(C^\bullet_{\mathrm{st}}(D'))$ is zero.*

Proof. i) By Lemma 1.2.8 D_1 and D_2 are potentially semistable. Let L/K be a finite Galois extension such that D, D_1 and D_2 are semistable over L and let $I_{L/K}$ be the inertia subgroup of $\mathrm{Gal}(L/K)$. Then $H^1(I_{L/K}, \mathcal{D}_{\mathrm{pst}}(D_1)) = 0$ and one has an exact sequence

$$0 \to \mathcal{D}_{\mathrm{pst}}(D_1)^{I_{L/K}} \to \mathcal{D}_{\mathrm{pst}}(D)^{I_{L/K}} \to \mathcal{D}_{\mathrm{pst}}(D_2)^{I_{L/K}} \to 0$$

of K_0^{nr}-vector spaces equipped with a semilinear action of $\mathrm{Gal}(K_0^{nr}/K_0)$. Taking invariants we obtain that the sequence $0 \to \mathcal{D}_{\mathrm{st}}(D_1) \to \mathcal{D}_{\mathrm{st}}(D) \to \mathcal{D}_{\mathrm{st}}(D_2) \to 0$ is exact. Thus

$$0 \to C^\bullet_{\mathrm{st}}(D_1) \to C^\bullet_{\mathrm{st}}(D) \to C^\bullet_{\mathrm{st}}(D_2) \to 0$$

is an exact sequence of complexes. Passing to cohomology we obtain i).

ii) In dimension 1, the effacability follows from the description of $H^1(C^\bullet_{\mathrm{st}}(D))$ in terms of extensions (see Proposition 1.4.4 ii)). Namely, let $\mathrm{cl}(x) \in H^1(C^\bullet_{\mathrm{st}}(D))$ and let $0 \to D \to D_x \to \mathcal{R}(K) \to 0$ be the extension associated to x. Then the image of $\mathrm{cl}(x)$ in $H^1(C^\bullet_{\mathrm{st}}(D_x))$ is zero.

Now prove that $D \mapsto H^i(C^\bullet_{\mathrm{st}}(D))$ is effacable in dimension 2. Let $x \in \mathcal{D}_{\mathrm{st}}(D)$ and let $\mathrm{cl}(x)$ denote the class of x in $H^2(C^\bullet_{\mathrm{st}}(D))$. Set $x_1 = Nx$, $x_2 = Nx_1, \ldots, x_m = Nx_{m-1}, Nx_m = 0$ and consider the filtered (φ, N, G_K)-module M' defined by

$$\begin{aligned} &M' = \mathcal{D}_{\mathrm{pst}}(D) \oplus K_0 e_1 \oplus K_0 e_2 \oplus \cdots \oplus K_0 e_{m+1}, \\ &Ne_i = e_{i+1} \quad \text{if } 1 \leqslant i \leqslant m \text{ and } Ne_{m+1} = 0, \\ &\varphi e_i = p^{-i} e_i + p^{-i} x_{i-1} \quad \text{if } 1 \leqslant i \leqslant m+1, \end{aligned}$$

$$\mathrm{Fil}^i M'_K = \begin{cases} F^i\mathcal{D}_{\mathrm{dR}}(D) \oplus Ke_1 \oplus Ke_2 \oplus \cdots \oplus Ke_{m+1} & \text{if } i \leqslant 0 \\ F^i\mathcal{D}_{\mathrm{dR}}(D) & \text{if } i > 0. \end{cases}$$

Then one has an exact sequence of filtered (φ, N, G_K)-modules

$$0 \to \mathcal{D}_{\mathrm{pst}}(D) \to M' \to M'' \to 0 \tag{4}$$

where $M'' = \overset{m+1}{\underset{i=1}{\oplus}} K_0\overline{e}_i$, $N\overline{e}_i = \overline{e}_{i+1}$, $\varphi(\overline{e}_i) = p^{-i}\overline{e}_i$ and

$$\mathrm{Fil}^i M''_K = \begin{cases} M''_K & \text{if } i \leqslant 0 \\ 0 & \text{if } i > 0. \end{cases}$$

By Proposition 1.2.9 the sequence (4) corresponds to an exact sequence of potentially semistable (φ, Γ)-modules

$$0 \to D \xrightarrow{\alpha} D' \xrightarrow{\beta} D'' \to 0$$

such that $M' = \mathcal{D}_{\mathrm{pst}}(D')$ and $M'' = \mathcal{D}_{\mathrm{pst}}(D'')$. Since $(p\,\varphi - 1)\,e_1 = x$, the image of $\mathrm{cl}(x)$ in $H^2(C^\bullet_{\mathrm{st}}(D'))$ is zero and the lemma is proved.

1.4.8. If D_1 and D_2 are two semistable modules we define a pairing

$$\cup \ : \ H^i(C^\bullet_{\mathrm{st}}(D_1)) \times H^j(C^\bullet_{\mathrm{st}}(D_2)) \to H^{i+j}(C^\bullet_{\mathrm{st}}(D_1 \otimes D_2))$$

by the formulas

$$\mathrm{cl}(x_1) \cup \mathrm{cl}(x_2) = \mathrm{cl}(x_1 \otimes x_2) \qquad \text{if } i = 0,\ j = 0,$$

$$\mathrm{cl}(x_1) \cup \mathrm{cl}(x_2 \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D_2)}, y_2, z_2) \ = \ \mathrm{cl}(x_1 \otimes x_2 \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D_1 \otimes D_2)}, x_1 \otimes y_2, x_1 \otimes z_2),$$
$$\text{if } i = 0,\ j = 1,$$

$$\mathrm{cl}(x_1) \cup \mathrm{cl}(x_2) = \mathrm{cl}(x_1 \otimes x_2) \qquad \text{if } i = 0,\ j = 2,$$

$$\mathrm{cl}(x_1 \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D_2)}, y_1, z_1) \cup \mathrm{cl}(x_2 \pmod{\mathrm{Fil}^0\mathcal{D}_{\mathrm{dR}}(D_2)}, y_2, z_2) = \mathrm{cl}(z_1 \otimes y_2 + y_1 \otimes p\varphi(z_2))$$
$$\text{if } i = 1,\ j = 1.$$

An easy computation shows that this product is compatible with connecting homomorphisms. Namely, if $0 \to D_1 \to D'_1 \to D''_1 \to 0$ is exact, then the diagrams

$$\begin{array}{ccc} H^i(C^\bullet_{\mathrm{st}}(D''_1)) \times H^j(C^\bullet_{\mathrm{st}}(D_2)) & \xrightarrow{\ \cup\ } & H^{i+j}(C^\bullet_{\mathrm{st}}(D''_1 \otimes D_2)) \\ \downarrow & & \downarrow \\ H^{i+1}(C^\bullet_{\mathrm{st}}(D_1)) \times H^j(C^\bullet_{\mathrm{st}}(D_2)) & \xrightarrow{\ \cup\ } & H^{i+j+1}(C^\bullet_{\mathrm{st}}(D_1 \otimes D_2)) \end{array}$$

commute.

Proposition 1.4.9. *There exists a unique natural transformation of cohomological functors* $h^* : H^*(C^\bullet_{\mathrm{st}}(D)) \to H^*(D)$ *satisfying the following properties:*

1) h^0 *and* h^1 *coincide with the maps* $H^0(C^\bullet_{\mathrm{st}}(D)) \simeq H^0(D)$ *and* $H^1(C^\bullet_{\mathrm{st}}(D)) \simeq H^1_{st}(D) \to H^1(D)$ *given by Proposition 1.4.4.*

2) h^* *is compatible with cup-products.*

Proof. Remark that h^0 and h^1 are already defined. Let $\mathrm{cl}(x) \in H^2(C^\bullet_{\mathrm{st}}(D))$. We define $h^2(\mathrm{cl}(x))$ by the usual way using effacability. Let

$$E_D \ :\ 0 \xrightarrow{\alpha} D \to D' \xrightarrow{\beta} D'' \to 0$$

be an exact sequence in $\mathbf{M}^{\varphi,\Gamma}_{\mathrm{pst},K}$ such that $\mathrm{cl}(x)$ vanishes in $H^2(C^\bullet_{\mathrm{st}}(D))$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(D') \arrow[r] & H^1(D'') \arrow[r, "\delta_1"] & H^2(D) \arrow[r] & H^2(D') \arrow[r] & \cdots \\
\cdots \arrow[r] & H^1(C^\bullet_{\mathrm{st}}(D')) \arrow[r] \arrow[u] & H^1(C^\bullet_{\mathrm{st}}(D'')) \arrow[r, "\Delta_1"] \arrow[u, "h^1"] & H^2(C^\bullet_{\mathrm{st}}(D)) \arrow[r] & H^2(C^\bullet_{\mathrm{st}}(D')) \arrow[r] & \cdots
\end{tikzcd}$$

Since the image of $\mathrm{cl}(x)$ in $H^2(C^\bullet_{\mathrm{st}}(D'))$ is zero, there exists $a \in H^1(C^\bullet_{\mathrm{st}}(D''))$ such that $\Delta_1(a) = \mathrm{cl}(x)$. Set $h^2(\mathrm{cl}(x)) = \delta_1 \circ h^1(a)$. It is well known (see for example [Sz]) that for abelian categories this gives a well defined morphism $H^2(C^\bullet_{\mathrm{st}}(D)) \to H^2(D)$. In our case this can be proved by the same way. Namely, for any morphism $f : D \to P$ of potentially semistable (φ,Γ)-modules the usual construction gives an exact sequence $f \circ E_D$ which seats in the diagram:

$$\begin{tikzcd}
E_D : & 0 \arrow[r] & D \arrow[r, "\alpha"] \arrow[d, "f"] & D' \arrow[r, "\beta"] \arrow[d] & D'' \arrow[r] \arrow[d, "="] & 0 \\
f \circ E_D : & 0 \arrow[r] & P \arrow[r] & P' \arrow[r] & D'' \arrow[r] & 0.
\end{tikzcd}$$

Indeed, here $P' = \mathrm{coker}(D \xrightarrow{(\alpha,f)} D' \oplus P)$ is a free $\mathcal{R}(K)$-module because D is saturated in $D' \oplus P$. Let $E'_D : 0 \to D \to X' \to X'' \to 0$ be another exact sequence such that $\mathrm{cl}(x)$ vanishes in $H^2(C^\bullet_{\mathrm{st}}(X'))$. Set $f_2 : D \oplus D \to P$, $f_2(d_1,d_2) = d_1 + d_2$ and consider $f_2 \circ (E_D \oplus E'_D)$. The injections $i_{1,2} : D \to D \oplus D$, $i_1(d) = (d,0)$, $i_2(d) = (0,d)$ induce natural morphisms $E_D \to f_2 \circ (E_D \oplus E'_D)$ and $E'_D \to f_2 \circ (E_D \oplus E'_D)$. This allows to show that $h_2(\mathrm{cl}(x))$ is well defined. A similar argument proves that h_2 is a morphism. The uniqueness of h_2 is clear. It remains to show that the diagrams

$$\begin{tikzcd}
H^i(C^\bullet_{\mathrm{st}}(D_1)) \times H^j(C^\bullet_{\mathrm{st}}(D_2)) \arrow[r, "\cup"] \arrow[d] & H^{i+j}(C^\bullet_{\mathrm{st}}(D_1 \otimes D_2)) \arrow[d] \\
H^i(D_1) \times H^j(D_2) \arrow[r, "\cup"] & H^{i+j}(D_1 \otimes D_2)
\end{tikzcd}$$

commute. It is immediate if $i = j = 0$ and the general case can be deduced using dimension shifting.

Corollary 1.4.10. *Let* D *be a potentially semistable* (φ,Γ)*-module. Then* $H^1_f(D^*(\chi))$ *is the orthogonal complement to* $H^1_f(D)$ *under the duality*

$$H^1(D) \times H^1(D^*(\chi)) \xrightarrow{\cup} H^2(\mathcal{R}(K)(\chi)) \simeq \mathbb{Q}_p.$$

Proof. By Proposition 1.4.9 we have a commutative diagram

$$\begin{array}{ccc} H^1(C^\bullet_{\mathrm{st}}(D))\times H^1(C^\bullet_{\mathrm{st}}(D^*(\chi))) & \xrightarrow{\cup} & H^2(C^\bullet_{\mathrm{st}}(\mathcal{R}(K)(\chi))) \\ \downarrow & & \downarrow \\ H^1(D)\times H^1(D^*(\chi)) & \xrightarrow{\cup} & \mathbb{Q}_p \end{array}$$

From the definition of the cup product it follows immediately that $x\cup y=0$ for all $x\in H^1(C^\bullet_{\mathrm{cris}}(D))$ and $y\in H^1(C^\bullet_{\mathrm{cris}}(D^*(\chi)))$. This proves that $H^1_f(D^*(\chi))$ and $H^1_f(D)$ are orthogonal to each other. Next, by Corollary 1.4.5 together with the Euler-Poincaré characteristic formula we have

$$\dim_{\mathbb{Q}_p} H^1_f(D)+\dim_{\mathbb{Q}_p} H^1_f(D^*(\chi))=\dim\mathcal{D}_{\mathrm{dR}}(D)+\dim_{\mathbb{Q}_p} H^0(D)+\dim_{\mathbb{Q}_p} H^0(D^*(\chi))=\dim_{\mathbb{Q}_p} H^1(D).$$

The corollary is proved.

1.5. Semistable modules of rank 1.

1.5.1. In this section we compute $H^1_f(D)$ of semistable modules of rank 1. For any continuous character $\delta:\mathbb{Q}_p^*\to\mathbb{Q}_p^*$ let $\mathcal{R}(K)(\delta)$ denote the (φ,Γ)-module $\mathcal{R}(K)e_\delta$ such that $\varphi(e_\delta)=\delta(p)e_\delta$ and $\gamma(e_\delta)=\delta(\chi(\gamma))\,e_\delta$, $\gamma\in\Gamma$. Write x for the character given by the identity map and $|x|$ for $|x|=p^{-v_p(x)}$.

Lemma 1.5.2. *The following statements are equivalent:*

i) D is a semistable module of rank 1;

ii) D is a crystalline module of rank 1;

iii) D is isomorphic to $\mathcal{R}(K)(\delta)$ where $\delta:\mathbb{Q}_p^*\to\mathbb{Q}_p^*$ is such that $\delta(u)=u^m$ ($u\in\mathbb{Z}_p^*$) for some $m\in\mathbb{Z}$.

Proof. Since the operator $N:\mathcal{D}_{\mathrm{st}}(\mathcal{R}(K)(\delta))\to\mathcal{D}_{\mathrm{st}}(\mathcal{R}(K)(\delta))$ is nilpotent, it is clear that *i)* $\Leftrightarrow$ *ii)* and *iii)* $\Rightarrow$ *i)*. We prove that *i)* $\Rightarrow$ *iii)*. If $\mathcal{R}(K)(\delta)$ is semistable it is a de Rham and there exists $n\in\mathbb{Z}$ such that $\delta(u)=u^m$ on $1+p^n\mathbb{Z}_p\subset\mathbb{Z}_p^*$. Replacing δ by δx^{-m} we may assume that $m=0$. As $(\mathcal{R}(K)[1/t])^{\Gamma_n}=K_0$, we obtain that $\mathcal{D}_{\mathrm{cris}}(\mathcal{R}(K)(\delta))=(K_0(\delta))^\Gamma$ is a one-dimensional L-vector space. Thus $\delta=0$ and the proposition is proved.

Proposition 1.5.3. *Let $\mathcal{R}(K)(\delta)$ be such that $\delta(u)=u^m$ ($u\in\mathbb{Z}_p^*$) for some $m\in\mathbb{Z}$. Then*

i)

$$H^0(\mathcal{R}(K)(\delta))=\begin{cases}\mathbb{Q}_p t^m & \text{if } \delta=x^{-m},\ k\in\mathbb{N}\\ 0 & \text{otherwise.}\end{cases}$$

ii) Assume that $m\geqslant 1$. If $\delta\neq|x|x^m$ then $H^1(\mathcal{R}(K)(\delta))=H^1_f(\mathcal{R}(K)(\delta))$ is a $\mathbb{Q}_p$-vector space of dimension $[K:\mathbb{Q}_p]$. If $\delta=|x|x^m$ then $H^1_{st}(\mathcal{R}(K)(\delta))=H^1(\mathcal{R}(K)(\delta))$ is a $\mathbb{Q}_p$-vector space of dimension $[K:\mathbb{Q}_p]+1$ and $\dim_{\mathbb{Q}_p}H^1_f(\mathcal{R}(K)(\delta))=[K:\mathbb{Q}_p]$.

iii) Assume that $m\leqslant 0$. If $\delta\neq x^m$, then $H^1(\mathcal{R}(K)(\delta))$ is a $\mathbb{Q}_p$-vector space of dimension $[K:\mathbb{Q}_p]$ and $H^1_f(\mathcal{R}(K)(\delta))=0$. If $\delta=x^m$, then $\dim_{\mathbb{Q}_p}H^1(\mathcal{R}(K)(\delta))=[K:\mathbb{Q}_p]+1$ and $\dim_{\mathbb{Q}_p}H^1_f(\mathcal{R}(K)(\delta))$ is the one dimensional $\mathbb{Q}_p$-vector space generated by $\mathrm{cl}(t^{-m},0)\otimes e_\delta$.

Proof. This is a direct application of the Euler-Poincaré caracteristic formula together with Corollary 1.4.5.

1.5.4. In the remainder of this paragraph we suppose that $K=\mathbb{Q}_p$ and write $\mathcal{R}$ for $\mathcal{R}(\mathbb{Q}_p)$. In [Cz4], Colmez studies general (φ,Γ)-modules of rank 1 over $\mathcal{R}$. He proves that any (φ,Γ)-module of rank 1 is isomorphic to $\mathcal{R}(\delta)$ for some $\delta:\mathbb{Q}_p^*\to\mathbb{Q}_p^*$ (see [Cz4], Proposition 3.1) and describes $H^1(\mathcal{R}(\delta))$ in details.

Proposition 1.5.5. *i) If $\delta = x^{-m}$, $m \geqslant 0$ then $\mathrm{cl}(t^m,0)\, e_\delta$ and $\mathrm{cl}(0,t^m)\, e_\delta$ form a L-basis of $H^1(\mathcal{R}(\delta))$.*
iv) If $\delta = |x|x^m$, $m \geqslant 1$, then $H^1(\mathcal{R}(\delta))$ is generated over $\mathbb{Q}_p$ by $\mathrm{cl}(\alpha_m)$ and $\mathrm{cl}(\beta_m)$ where

$$\alpha_m = \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}, a\right)\, e_\delta, \qquad (1-\varphi)\,a = (1-\chi(\gamma)\gamma)\,\left(\frac{1}{\pi}+\frac{1}{2}\right),$$
$$\beta_m = \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(b, \frac{1}{\pi}\right)\, e_\delta, \qquad (1-\varphi)\,\left(\frac{1}{\pi}\right) \,=\, (1-\chi(\gamma)\,\gamma)\,b$$

and ∂ denotes the differential operator $(1+\pi)\,d/d\pi$.

Proof. See sections 2.3-2.5 of [Cz4].

To simplify notations we will write e_m (respectively w_m) for e_δ if $\delta = |x|x^m$, $m \geqslant 1$ (respectively if $\delta = x^{-m}$, $m \geqslant 0$).

Corollary 1.5.6. *Any element $\mathrm{cl}((f,g)e_m) \in H^1(\mathcal{R}(x^m|x|))$ can be written in the form*

$$\mathrm{cl}\,((f,g)e_m) \,=\, x\,\mathrm{cl}(\alpha_m) \,+\, y\,\mathrm{cl}(\beta_m)$$

where $x = \mathrm{res}(ft^{m-1}dt)$, $y = \mathrm{res}(gt^{m-1}dt)$.

Proof. The case $m = 1$ is implicitly contained in the proof of Proposition 2.8 of [Cz4] and the general case can be proved by the same argument. Namely, the formulas

$$\mathrm{res}(\chi(\gamma)\gamma(h)dt) \,=\, \mathrm{res}(hdt), \qquad \mathrm{res}(\varphi(h)dt) \,=\, \mathrm{res}(hdt)$$

imply that

$$\mathrm{res}(\chi(\gamma)^m\gamma(h)\,t^{m-1}dt) \,=\, \mathrm{res}(ht^{m-1}dt), \qquad \mathrm{res}(p^{m-1}\varphi(h)t^{m-1}dt) \,=\, \mathrm{res}(ht^{m-1}dt).$$

Thus $\mathrm{res}(ft^{m-1}dt)$ and $\mathrm{res}(gt^{m-1}dt)$ do not change if we add to $(f,g)e_m$ a boundary $((1-\varphi)\,(he_m), (1-\gamma)\,(he_m))$. As $a \in \mathcal{R}^+ = \mathcal{R}\cap\mathbb{Q}_p[[\pi]]$ (see [Ben1], Lemma 2.1.3 or [Cz4]), we have $\mathrm{res}(\alpha_m t^{m-1}dt) \,=\, (1,0)$. For any $g \in \mathbf{A}^+_{\mathbb{Q}_p}$ set

$$\ell(g) \,=\, \frac{1}{p}\log\left(\frac{g^p}{\varphi(g)}\right).$$

As $g^p/\varphi(g) \equiv 1 \pmod p$ in $\mathbf{A}_{\mathbb{Q}_p}$, it is easy to see that $\ell(g) \in \mathbf{A}_{\mathbb{Q}_p}$. Let ψ denote the operator $\psi(G(\pi)) = \dfrac{1}{p}\varphi^{-1}\displaystyle\sum_{\zeta^p=1} G((1+\pi)\zeta-1)$. A short computation shows that $\psi(\ell(\pi)) = 0$. Then by Lemma 1.5.1 of [CC] there exists a unique $b_0 \in \mathbf{A}^\dagger_{\mathbb{Q}_p}$ such that $(\gamma-1)\,b_0 \,=\, \ell(\pi)$. Applying the operator ∂ to this formula we obtain that $b = \partial b_0$. Thus $\mathrm{res}((\partial^{m-1}b)t^{m-1}dt) \,=\, 0$ and $\mathrm{res}(\beta_m t^{m-1}dt) \,=\, (0,1)$. By Proposition 1.5.5 any $\mathrm{cl}((f,g)w_m) \in H^1(\mathcal{R}_L(|x|x^m)$ can be written in the form $\mathrm{cl}\,((f,g)w_m) \,=\, x\,\mathrm{cl}(\alpha_m) + y\,\mathrm{cl}(\beta_m)$ Applying previous formulas we obtain that $x = \mathrm{res}(ft^{m-1}dt)$ and $y = \mathrm{res}(gt^{m-1}dt)$. The corollary is proved.

1.5.7. Set

$$\alpha_m^* = \left(1-\frac{1}{p}\right)\,\mathrm{cl}(\alpha_m), \quad \beta_m^* = \left(1-\frac{1}{p}\right)\log\chi(\gamma)\,\mathrm{cl}(\beta_m) \qquad \text{if } m \geqslant 1,$$
$$x_m^* = \mathrm{cl}(t^m,0)w_m, \quad y_m^* = \log\chi(\gamma)\,\mathrm{cl}(0,t^m)w_m \qquad \text{if } m \geqslant 0.$$

If $m = 0$, then $H^1(\mathcal{R}_L(x^m) = H^1(\mathcal{R}))$ is canonically isomorphic to $H^1(\mathbb{Q}_p,\mathbb{Q}_p) = \mathrm{Hom}(G_{\mathbb{Q}_p},\mathbb{Q}_p)$. Let $\mathrm{ord} : \mathrm{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \to \mathbb{Z}_p$ denote the character determined by $\mathrm{ord}(\varphi) = 1$. Then ord and $\log\chi$ form a canonical basis of $\mathrm{Hom}(G_{\mathbb{Q}_p},\mathbb{Q}_p)$ which corresponds to x_0^*, y_0^*. If $m = 1$, then $H^1(\mathcal{R}(|x|x)) \simeq H^1(\mathcal{R}(\chi))$ is isomorphic to $H^1(\mathbb{Q}_p,\mathbb{Q}_p(1))$. Let $\kappa : \mathbb{Q}_p^* \to H^1(\mathbb{Q}_p,\mathbb{Q}_p(1))$ denote the Kummer map. Then $\kappa(u) = -\log(u)\alpha_1^*$ if $u \equiv 1 \pmod p$ and $\kappa(p) = -\beta_1^*$ (see [Ben1], Proposition 2.1.5).

Proposition 1.5.8. *i) Let $m \geqslant 1$. The basis α_m^*, β_m^* is dual to the basis y_m^*, x_m^* under the pairing*

$$H^1(\mathcal{R}(|x|x^m)) \times H^1(\mathcal{R}(x^{1-m})) \xrightarrow{\cup} \mathbb{Q}_p.$$

In particular, α_m^* generates $H^1_f(\mathcal{R}(|x|x^m))$.
ii) The canonical isomorphism

$$H^1(\mathcal{R}(|x|x^m)) \xrightarrow{\sim} H^1(C^\bullet_{\mathrm{st}}(\mathcal{R}(|x|x^m)))$$

sends α_m^* to $-\mathrm{cl}(1,0,0)$ and β_m^* to $-\mathrm{cl}(0,0,1)$.

Proof. i) The proof follows from the construction of the cup product (see 1.1.5) together with the explicit description of the isomorphism $H^2(\mathcal{R}(\chi)) \simeq \mathbb{Q}_p$ reviewed in 1.1.7. Remark that

$$\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right) \equiv \frac{(-1)^{m-1}(m-1)!}{t^m} \pmod{\mathbb{Q}_p[[\pi]]}.$$

Thus

$$\alpha_m^* \cup y_m^* = \frac{(-1)^{m-1}}{(m-1)!}\mathrm{res}\left(\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)t^{m-1}\,\frac{d\pi}{1+\pi}\right) = 1.$$

On the other hand it is clear that $\alpha_m^* \cup x_m^* = 0$. The proof of other formulas is analogous.

iia) Recall that e_m denote the canonical basis of $\mathcal{R}(|x|x^m)$. Let

$$0 \longrightarrow \mathcal{R}(|x|x^m) \longrightarrow D \longrightarrow \mathcal{R} \longrightarrow 0$$

be the extension associated to α_m. Then $D = \mathcal{R}(|x|x^m) + \mathcal{R}u_m$ where

$$(\varphi-1)\,u_m \;=\; \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{\pi}+\frac{1}{2}\right)e_m, \qquad (\gamma-1)\,u_m \;=\; \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}(a)\,e_m.$$

As $\mathrm{cl}(\alpha_m) \in H^1_f(\mathcal{R}(|x|x^m)$, we have an exact sequence

$$0 \longrightarrow \mathcal{D}_{\mathrm{cris}}(\mathcal{R}(|x|x^m)) \longrightarrow \mathcal{D}_{\mathrm{cris}}(D) \longrightarrow \mathcal{D}_{\mathrm{cris}}(\mathcal{R}) \longrightarrow 0 \tag{5}$$

which induces an isomorphism

$$\mathcal{D}_{\mathrm{cris}}(D)^{\varphi=1} \simeq \mathcal{D}_{\mathrm{cris}}(\mathcal{R})^{\varphi=1}.$$

Then there exists a unique $c_m e_m \in \mathcal{R}(|x|x^m)\,[1/t]$ such that

$$(\varphi-1)\,(u_m + c_m e_m) \;=\; 0.$$

Thus $(1-p^{m-1}\varphi)\,c_m \;=\; -\dfrac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\dfrac{1}{\pi}+\dfrac{1}{2}\right)$ and a short computation shows that $c_m = \dfrac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}c_1$. For $m=1$ this equation can be written in the form

$$(1-\varphi/p)\,(tc_1) = -t\left(\frac{1}{\pi}+\frac{1}{2}\right) \in \mathcal{R}^+,$$

where $\mathcal{R}^+ = \mathcal{R} \cap \mathbb{Q}_p[[\pi]]$ and $t\left(\frac{1}{\pi}+\frac{1}{2}\right) \equiv 1 \pmod{\pi^2}$. Then Lemma A.1 of [Cz4] implies that $tc_1 \in \mathcal{R}^+$ and therefore satisfies $tc_1 \equiv -\left(1-\frac{1}{p}\right)^{-1} \pmod{\pi}$. Taking derivations we obtain by induction that $t^m c_m \equiv -\left(1-\frac{1}{p}\right)^{-1} \pmod{\pi}$. Now from the proof of Proposition 1.4.4 it follows that the extension (5) corresponds to the class $-\left(1-\frac{1}{p}\right)^{-1} \mathrm{cl}(1,0,0) \in H^1(C^\bullet_{\mathrm{st}}(\mathcal{R}(|x|x^m)))$ and the first formula of ii) is proved.

iib) Let $0 \longrightarrow \mathcal{R}(|x|x^m) \longrightarrow D \longrightarrow \mathcal{R} \longrightarrow 0$ be the extension associated to $\mathrm{cl}(\beta_m)$. Write $D = \mathcal{R}(|x|x^m) + \mathcal{R}v_m$ where

$$(\varphi-1)\,v_m \;=\; \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}(b)\,e_m, \qquad (\gamma-1)\,v_m \;=\; \frac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{\pi}\right)\,e_m.$$

By Proposition 1.5.3 D is semistable and there exists $d_m e_m \in \mathcal{R}_{\log}(|x|x^m)[1/t]$ such that $(\gamma-1)\,(v_m + d_m e_m) \;=\; 0$. Thus

$$(\chi(\gamma)^m\gamma - 1)\,d_m \;=\; \frac{(-1)^m}{(m-1)!}\,\partial^{m-1}\left(\frac{1}{\pi}\right).$$

An easy computation shows that $d_m \;=\; \dfrac{(-1)^{m-1}}{(m-1)!}\,\partial^{m-1}\,(d_1)$. Set $\nabla \;=\; \log(\gamma)/\log\chi(\gamma)$ and $\nabla_0 \;=\; \nabla/(\gamma-1)$. As $\nabla \;=\; t\partial$ on $\mathcal{R}$ we see that one can take $d_1 \;=\; -t^{-1}\nabla_0(\log\pi) + (\chi(\gamma)-1)^{-1}$. From

$$(\varphi-1)\,d_1 = t^{-1}\nabla_0\left(1-\frac{\varphi}{p}\right)\log\pi,$$

it follows that

$$(\gamma-1)\,\left((\varphi-1)\,d_1e_1\right) \;=\; \nabla\left(1-\frac{\varphi}{p}\right)(\log\pi)e_1 = (1-\varphi)\,\partial(\log\pi)e_1 = (1-\varphi)\left(\frac{1}{\pi}\right)e_1.$$

As $\psi(tb) = \dfrac{1}{p}\psi(\varphi(t)b) = t\psi(b)/p = 0$, the element $bt \in \mathcal{R}^{\psi=0}$ is a solution of the equation $(\gamma-1)\,x = (1-\varphi)(1/\pi)$. On the other hand, as $\prod\limits_{\zeta^p=1}((1+X)\,\zeta-1) = \varphi(X)$, we have

$$\psi\log\left(\frac{\varphi(X)}{X^p}\right) \;=\; \frac{1}{p}\,\varphi^{-1}\log\left(\frac{\varphi(X)^p}{\prod\limits_{\zeta^p=1}((1+X)\,\zeta-1)^p}\right) \;=\; 0.$$

Thus $t(\varphi-1)d_1 = (1-\varphi)\nabla_0\,(1-1/p)\,\log\pi \in \mathcal{R}^{\psi=0}$ is also a solution of $(\gamma-1)\,x = (1-\varphi)(1/\pi)$. As $\gamma-1$ is bijective on $\mathcal{R}^{\psi=0}$ (see for example [Ber3], Lemma I.3) this implies that $(\varphi-1)\,d_1 = -b$ and $(\varphi-1)\,(d_me_m) = \frac{(-1)^m}{(m-1)!}\,\partial^{m-1}(b)\,e_m$. Using the formula $\partial\,\gamma = \chi(\gamma)\,\gamma\,\partial$ we obtain

$$\partial^{m-1}\left(t^{-1}\nabla_0(\log\pi)\right) \;=\; (-1)^{m-1}(m-1)!\,t^{-m}\nabla_0(\log\pi) + t^{1-m}x,$$

where $x \in \pi^{1-m}\mathcal{R}^+$. Set $\gamma_1 = \gamma^{p-1}$. Then

$$\nabla_0 = \left(\sum_{i=0}^{p-2}\gamma^i\right)\frac{\nabla}{\gamma_1-1} \;=\; (\log\chi(\gamma_1))^{-1}\left(\sum_{i=0}^{p-2}\gamma^i\right)\sum_{n=0}^{\infty}(-1)^n\frac{(\gamma_1-1)^n}{n+1}.$$

Remark that

$$\left(\sum_{i=0}^{p-2}\gamma^i\right)\log\pi=(p-1)\log\pi+y_1$$

where

$$y_1=\sum_{i=1}^{p-2}(\gamma^i-1)\log\pi=\log\left(\frac{\pi^{1+\gamma+\dots+\gamma^{p-2}}}{\pi^{p-1}}\right).$$

In particular,

$$\iota_1(y_1)(0)=\log\left(\frac{N_{\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p}(\zeta_p-1)}{(\zeta_p-1)^{p-1}}\right)\;=\;\log\left(\frac{p}{(\zeta_p-1)^{p-1}}\right)\;=\;-(p-1)\log(\zeta_p-1).$$

Set

$$y_2=\left(\sum_{i=0}^{p-2}\gamma^i\right)\sum_{n=1}^{\infty}(-1)^n\frac{(\gamma_1-1)^n}{n+1}\log\pi.$$

As $\iota_1((\gamma_1-1)\log\pi)(0)=0$ we have $\iota_1(y_2)(0)=0$ and

$$d_m=-\left(\log\chi(\gamma)\right)^{-1}t^{-m}(\log\pi+y)$$

where $y\in\mathbf{B}^{\dagger,p-1}_{\mathrm{rig},\mathbb{Q}_p}$ is such that $\iota_1(y)=-\log(\zeta_p-1)$. Remark that as usually the p-adic logarithm is normalized by $\log(p)=0$. Thus

$$\mathrm{cl}(\beta_m)\;=\;(\log\chi(\gamma))^{-1}\left(1-\frac{1}{p}\right)t^{-m}\mathrm{cl}\left((p^{-1}\varphi-1)(\log\pi+y),(\gamma-1)(\log\pi+y)\right)e_m. \tag{6}$$

Let $G(|x|,m)$ and $G'(|x|,m)$ be two elements of $\mathbf{B}^{\dagger,p-1}_{\mathrm{rig},\mathbb{Q}_p}$ satisfying the following conditions:

$$\iota_n(G(|x|,m))\equiv p^{-n}\pmod{t^m},\quad \iota_n(G'(|x|,m))\equiv\log(\zeta_{p^n}e^{t/p^n}-1)\pmod{t^m},\quad\forall n\geqslant 1.$$

By [Cz4], Proposition 2.19 there exist unique $\lambda,\mu\in\mathbb{Q}_p$ such that $\beta_m^*=\mathrm{cl}(a,b)e_m$ where

$$\begin{aligned}a&=t^{-m}(p^{-1}\varphi-1)(\lambda G(|x|,m)+\mu(\log\pi-G'(|x|,m))),\\ b&=t^{-m}(\gamma-1)(\lambda G(|x|,m)+\mu(\log\pi-G'(|x|,m))).\end{aligned} \tag{7}$$

Remark that if two cocycles $t^{-m}((p^{-1}\varphi-1)x,(\gamma-1)x)$ and $t^{-m}((p^{-1}\varphi-1)y,(\gamma-1)y)$ $(x,y\in\mathcal{R}_{\log})$ are homologous in $Z^1(\mathcal{R}(|x|x^m)$, then $y=x+t^mz$ for some $z\in\mathcal{R}$. In particular, if $x,y\in\mathbf{B}^{\dagger,p-1}_{\mathrm{rig},\mathbb{Q}_p}$ then $\iota_1(x)(0)=\iota_1(y)(0)$. Comparing (6) and (7) we find that $\lambda=0$ and $\mu=(\log\chi(\gamma))^{-1}$. Therefore there exists a lifting $v_m'\in D$ of $1\in\mathcal{R}$ such that the (φ,N)-module $\mathcal{D}_{\mathrm{st}}(D)=(D\otimes_{\mathcal{R}}\mathcal{R}_{\log}[1/t])^{\Gamma}$ is generated by $f_1=t^{-m}e_m$ and $f_2=v_m'+(\log\chi(\gamma))^{-1}(\log\pi-G'(|x|,m))f_1$. We see immediately that $\varphi(f_2)=f_2$ and $N(f_2)=-(1-1/p)^{-1}(\log\chi(\gamma))^{-1}f_1$ and $f_2\in\mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(D)$. Thus D corresponds to the class $(1-1/p)^{-1}(\log\chi(\gamma))^{-1}\mathrm{cl}(0,0,-1)\in H^1(C^{\bullet}_{\mathrm{st}}(\mathcal{R}(|x|x^m))$. The proposition is proved.

Proposition 1.5.9. *Let D be a semistable (φ,Γ)-module of rank d with Hodge-Tate weights $k_1,\dots,k_d$. Assume that $\mathcal{D}_{\mathrm{st}}(D)=\mathcal{D}_{\mathrm{st}}(D)^{\varphi=\lambda}$ for some $\lambda\in\mathbb{Q}_p$. Then*

$$D\simeq \overset{d}{\underset{i=1}{\oplus}}\mathcal{R}(\delta_i),$$

where δ_i are defined by $\delta_i(u)=u^{-k_i}$ $(u\in\mathbb{Z}_p^*)$ and $\delta_i(p)=\lambda p^{-k_i}$.

Proof. 1) We prove the Proposition by induction on $d=\mathrm{rg}(D)$. The case $d=1$ is trivial. Let D be a semistable (φ,Γ)-module of rank 2 with Hodge weights $k_1\leqslant k_2$. Choose a non-zero $v\in \mathrm{Fil}^{k_2}\mathcal{D}_{\mathrm{st}}(D)$ and put $\mathcal{F}_1\mathcal{D}_{\mathrm{st}}(D)=\mathbb{Q}_pv$, $\mathcal{F}_2\mathcal{D}_{\mathrm{st}}(D)=D$. By Proposition 1.3.2 the triangulation of D associated to this filtration gives rise to an exact sequence

$$0\longrightarrow\mathcal{R}(\delta_2)\longrightarrow D\longrightarrow\mathcal{R}(\delta_1)\longrightarrow 0$$

where $\delta_i(p)=\lambda p^{-k_i}$ and $\delta_i(u)=u^{-k_i}$, $u\in\mathbb{Z}_p^*$. Put $\delta=\delta_2\delta_1^{-1}$. Then $\delta(x)=x^{-k}$ with $k=k_2-k_1\geqslant 0$ and $D(\delta_1^{-1})$ can be viewed as a semistable extension

$$0\longrightarrow\mathcal{R}(x^{-k})\longrightarrow D(\delta_1^{-1})\longrightarrow\mathcal{R}\longrightarrow 0.$$

The relation $N\varphi=p\,\varphi N$ together with the fact that $\mathcal{D}_{\mathrm{st}}(D)$ is pure implies that $\mathcal{D}_{\mathrm{st}}(D)^{N=0}=\mathcal{D}_{\mathrm{st}}(D)$. Thus D is crystalline and by Proposition 1.5.3 the class of $D(\delta^{-1})$ in $H^1(\mathcal{R}(\delta))$ is ax_k^* for some $a\in\mathbb{Q}_p$. Write $D(\delta_1^{-1})=\mathcal{R}e_\delta\oplus\mathcal{R}e$ where $e\in D$ is the lifting of $1\in\mathcal{R}$ such that $\gamma(e)=e$. Then $m_1=t^ke_\delta$ and $m_2=e$ form a basis of $\mathcal{D}_{\mathrm{cris}}(D(\delta_1^{-1}))$ and $\varphi(m_2)=m_2-am_1$. On the other hand, φ acts trivially on $\mathcal{D}_{\mathrm{st}}(D(\delta_1^{-1}))$ and we obtain that $a=0$ and $D(\delta_1^{-1})\simeq\mathcal{R}(\delta)\oplus\mathcal{R}$.

2) Now assume that the proposition holds for (φ,Γ)-modules of rank $d-1$. Let D be a pure semistable module of rank d with Hodge weights $k_1\leqslant k_2\leqslant\ldots\leqslant k_d$. Choose a non zero $v\in\mathrm{Fil}^{k_d}\mathcal{D}_{\mathrm{st}}(D)$ and consider the submodule of D which corresponds to $\mathbb{Q}_pv$ by Proposition 1.2.9:

$$\mathcal{R}(\delta_d)=D\cap(\mathcal{R}_{\log}[1/t]v).$$

Then $\delta_d(p)=\lambda p^{-k_d}$, $\delta_d(u)=u^{-k_d}$ $(u\in\mathbb{Z}_p^*)$ and we have an exact sequence

$$0\longrightarrow\mathcal{R}(\delta_d)\longrightarrow D\longrightarrow D'\longrightarrow 0$$

where D' is semistable of rank $d-1$ and such that $\mathcal{D}_{\mathrm{st}}(D')^{\varphi=\lambda}=\mathcal{D}_{\mathrm{st}}(D')$. Then $D'\simeq\overset{d-1}{\underset{i=1}{\oplus}}\mathcal{R}(\delta_i)$ and

$$\mathrm{Ext}^1(D',\mathcal{R}(\delta_d))\simeq\overset{d-1}{\underset{i=1}{\oplus}}\mathrm{Ext}^1(\mathcal{R}(\delta_i)),\mathcal{R}(\delta_d)).$$

Let x denote the class of D in $\mathrm{Ext}^1(D',\mathcal{R}(\delta_d))$. Since x is semistable, for any i its image x_i in $\mathrm{Ext}^1(\mathcal{R}(\delta_i)),\mathcal{R}(\delta_d))$ is semistable too. From 1) it follows that $x_i=0$. Thus $x=0$ and $D\simeq D'\oplus\mathcal{R}(\delta_d)$. The proposition is proved.

1.5.10. Let D be a semistable module. Assume that $\mathcal{D}_{\mathrm{st}}(D)=\mathcal{D}_{\mathrm{st}}(D)^{\varphi=1}$. By Proposition 1.5.9 D is crystalline and

$$D\simeq\overset{d}{\underset{i=1}{\oplus}}\mathcal{R}(x^{k_i}),\qquad k_i\leqslant 0.$$

In particular $\mathcal{D}_{\mathrm{st}}(D)=D^\Gamma$ and the map

$$i_D\,:\,\mathcal{D}_{\mathrm{st}}(D)\times\mathcal{D}_{\mathrm{st}}(D)\longrightarrow H^1(D)\tag{8}$$

given by $i_D(\alpha,\beta)=\mathrm{cl}(\alpha,\beta)$ is an isomorphism. Now, if $\mathcal{D}_{\mathrm{st}}(D)^{\varphi=p^{-1}}=\mathcal{D}_{\mathrm{st}}(D)$ then $\mathcal{D}_{\mathrm{st}}(D^*(\chi))^{\varphi=1}=\mathcal{D}_{\mathrm{st}}(D^*(\chi))$ and we define i_D by duality. Let $i_{D,f}$ and $i_{D,c}$ denote the restriction of i_D on the first and the second direct summand respectively. Then $\mathrm{Im}(i_{D,f})=H^1_f(D)$. Set $H^1_c(D)=\mathrm{Im}(i_{D,c})$.

§2. The $\mathcal{L}$-invariant

2.1. Definition of the $\mathcal{L}$-invariant.

2.1.1. In this section we generalize Greenberg's definition of the $\mathcal{L}$-invariant. Fix a finite set of primes S and denote by $\mathbb{Q}^{(S)}/\mathbb{Q}$ the maximal Galois extension of $\mathbb{Q}$ unramified outside $S\cup\{\infty\}$. Set $G_S=\mathrm{Gal}(\mathbb{Q}^{(S)}/\mathbb{Q})$. If M is a topological G_S-module, we denote by $H^*_S(M)$ the continuous cohomology of G_S with coefficients in M. A p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is said to be pseudo-geometric if it satisfies the following conditions:

1) There exists a finite S such that V is unramified outside $S\cup\{\infty\}$.

2) V is potentially semi-stable at p.

Let V be a pseudo-geometric representation. Following Bloch and Kato [BK], for any finite place v we define a subgroup $H^1_f(\mathbb{Q}_v,V)$ of $H^1(\mathbb{Q}_v,V)$ by

$$H^1_f(\mathbb{Q}_v,V)=\begin{cases}\ker(H^1(\mathbb{Q}_v,V)\to H^1(\mathbb{Q}_v^{\mathrm{ur}},V)) & \text{if } v\neq p\\ \ker(H^1(\mathbb{Q}_v,V)\to H^1(\mathbb{Q}_p,V\otimes_{\mathbb{Q}_p}\mathbf{B}_{\mathrm{cris}})) & \text{if } v=p.\end{cases}$$

The Selmer group $H^1_f(V)$ of V is defined as

$$H^1_f(V)\;=\;\ker\left(H^1_S(V)\to\bigoplus_{v\in S}\frac{H^1(\mathbb{Q}_v,V)}{H^1_f(\mathbb{Q}_v,V)}\right).$$

Using the inflation-restriction sequence, it is easy to see that this definition does not depend on the choice of S. From the Poitou-Tate exact sequence and the ortogonality of $H^1_f(\mathbb{Q}_v,V)$ and $H^1_f(\mathbb{Q}_v,V^*(1))$ one obtains the following exact sequence which relates $H^1_f(V)$ and $H^1_f(V^*(1))$:

$$(9)\quad 0\to H^1_f(V)\to H^1_S(V)\to\bigoplus_{v\in S}\frac{H^1(\mathbb{Q}_v,V)}{H^1_f(\mathbb{Q}_v,V)}\to H^1_f(V^*(1))^*\to$$
$$H^2_S(V)\to\underset{v\in S}{\oplus}H^2(\mathbb{Q}_v,V)\to H^0_S(V^*(1))^*\to 0$$

(see [FP], Proposition 2.2.1). Together with the well known formula for the Euler characteristic this implies that

$$(10)\quad \dim_{\mathbb{Q}_p}H^1_f(V)-\dim_{\mathbb{Q}_p}H^1_f(V^*(1))-\dim_{\mathbb{Q}_p}H^0_S(V)+\dim_{\mathbb{Q}_p}H^0_S(V^*(1))=$$
$$\dim_{\mathbb{Q}_p}t_V(\mathbb{Q}_p)-\dim_{\mathbb{Q}_p}H^0(\mathbb{R},V).$$

2.1.2. Assume that V satisfies the following conditions:

C1) $H^1_f(V)=H^1_f(V^*(1))=0$.

C2) $H^0_S(V)=H^0_S(V^*(1))=0$.

C3) V is semistable at p and $\varphi\,:\,\mathbf{D}_{\mathrm{st}}(V)\to\mathbf{D}_{\mathrm{st}}(V)$ is semisimple at 1 and p^{-1}.

C4) The (φ,Γ)-module $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ has no crystalline subquotient of the form

$$0\to\mathcal{R}(|x|x^k)\to U\to\mathcal{R}\to 0,\qquad k\geqslant 1.$$

Set $d_{\pm}(V)=\dim_{\mathbb{Q}_p}(V^{c=\pm 1})$, where c denotes the complex conjugation. Comparing **C1-2)** with (10) we obtain that $\dim_{\mathbb{Q}_p}t_V(\mathbb{Q}_p)=d_+(V)$.

Question 2.1.3. Let V be an irreducible pseudo geometric representation which is semistable at p. Does it satisfy **C4)**? This is a straightforward generalization of the hypothesis **U** of [G]. If V is the

p-adic realization of a pure motive over $\mathbb{Q}$ of weight w then conjecturally $\varphi\,:\,\mathbf{D}_{\mathrm{cris}}(V)\rightarrow\mathbf{D}_{\mathrm{cris}}(V)$ is semisimple and all of its eigenvalues have the same complex absolute value. In this case **C4**) holds automatically and 1 and $1/p$ can not be eigenvalues of φ simultaneously.

2.1.4. We say that a (φ,N)-submodule D of $\mathbf{D}_{\mathrm{st}}(V)$ is admissible if the canonical projection $D\rightarrow t_V(\mathbb{Q}_p)$ is an isomorphism. To any admissible D we associate an increasing filtration $(D_i)_{i=-2}^2$ on $\mathbf{D}_{\mathrm{st}}(V)$ by

$$D_i\,=\,\begin{cases} 0 & \text{if } i=-2,\\ (1-p^{-1}\varphi^{-1})\,D+N(D^{\varphi=1}) & \text{if } i=-1,\\ D & \text{if } i=0,\\ D+\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\cap N^{-1}(D^{\varphi=p^{-1}}) & \text{if } i=1,\\ \mathbf{D}_{\mathrm{st}}(V) & \text{if } i=2.\end{cases}$$

Lemma 2.1.5. *i) $(D_i)_{i=-2}^2$ is the unique filtration on $\mathbf{D}_{\mathrm{st}}(V)$ by (φ,N)-submodules such that*

D1) *$D_{-2}=0$, $D_0=D$ and $D_2=\mathbf{D}_{\mathrm{st}}(V)$;*
D2) *$(\mathbf{D}_{\mathrm{st}}(V)/D_1)^{\varphi=1,N=0}=0$ and $D_{-1}=(1-p^{-1}\varphi^{-1})D_{-1}+N(D_{-1})$;*
D3) *$(D_0/D_{-1})^{\varphi=p^{-1}}=D_0/D_{-1}$ and $(D_1/D_0)^{\varphi=1}=D_1/D_0$.*

ii) Consider the canonical isomorphism $\mathbf{D}_{\mathrm{st}}(V^*(1))\simeq\mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{D}_{\mathrm{st}}(V),\mathbb{Q}_p)$ where the action of φ on the right hand side is given by $(\varphi(f))\,(x)\,=\,p^{-1}f(\varphi^{-1}(x))$. Set

$$D^*\,=\,\mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{D}_{\mathrm{st}}(V)/D,\mathbb{Q}_p).$$

Then D^* is admissible and $D^*_i=\mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{D}_{\mathrm{st}}(V)/D_{-i},\mathbb{Q}_p)$.

Proof. i) Since φ is semisimple at 1 and p^{-1}, we have a decomposition of D into a direct sum of φ-modules

$$D\simeq X\oplus D^{\varphi=1}\oplus D^{\varphi=p^{-1}}.$$

As $N\varphi=p\,\varphi N$, one has $N(D^{\varphi=1})\subset D^{\varphi=p^{-1}}$, $N(D^{\varphi=p^{-1}})\subset X$ and $NX\subset X$. In particular, $D_{-1}=X\oplus D^{\varphi=1}\oplus N(D^{\varphi=1})$ is a (φ,N)-module and $(D/D_{-1})^{\varphi=p^{-1}}=D/D_{-1}$. Next, as $1-p^{-1}\varphi^{-1}$ is bijective on X one has

$$(1-p^{-1}\varphi^{-1})D_{-1}+N(D_{-1})=X+D^{\varphi=1}+N(D^{\varphi=1})=D_{-1}.$$

Further, $D_1\,=\,X\oplus\Big(\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\cap N^{-1}(D^{\varphi=p^{-1}})\Big)\oplus D^{\varphi=p^{-1}}$. From this decomposition it follows immediately that D_1 is a (φ,N)-module such that $(D_1/D_0)^{\varphi=1}\,=\,D_1/D_0$. Now let $N(\bar x)\,=\,0$ for some $\bar x=x+D_1\in(\mathbf{D}_{\mathrm{st}}(V)/D_1)^{\varphi=1}$. As φ is semisimple at 1, we can assume that $x\in\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$. Then $Nx\in D^{\varphi=p^{-1}}$ and we obtain that $x\in\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\cap N^{-1}(D^{\varphi=p^{-1}})$. Thus $\bar x=0$ and $(\mathbf{D}_{\mathrm{st}}(V)/D_1)^{\varphi=1,N=0}=0$.

Conversely, assume that $(D_i)_{i=-2}^2$ is a filtration which satisfies **D1-3)**. From the semisimplicity of $\varphi-1$ it follows that $D_1=D+Y$ where $Y\subset\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$. Since $N(Y)\subset\mathbf{D}_{\mathrm{st}}(V)^{\varphi=p^{-1}}\cap D_1=D^{\varphi=p^{-1}}$, one has $Y\subset\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\cap N^{-1}(D^{\varphi=p^{-1}})$. Let $\bar x=x+D_1$, $x\in\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}$. We showed that $N(\bar x)=0$ if and only if $x\in N^{-1}(D^{\varphi=p^{-1}})$ and the condition **D2)** implies that $Y=\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\cap N^{-1}(D^{\varphi=p^{-1}})$. Thus $D_1=D+\mathbf{D}_{\mathrm{st}}(V)^{\varphi=1}\cap N^{-1}(D^{\varphi=p^{-1}})$. A similar argument shows that $D_{-1}=(1-p^{-1}\varphi^{-1})D+N(D^{\varphi=1})$.

ii) The second statement follows from the uniqueness proved in i) and the fact that the filtration $\mathrm{Hom}_{\mathbb{Q}_p}(\mathbf{D}_{\mathrm{st}}(V)/D_{-i},\mathbb{Q}_p)$ satisfies **D1-3)**.

2.1.6. Let D be an admissible (φ,N)-module and $(D_i)_{i=-2}^2$ the associated filtration. By Proposition 1.2.9 it induces a filtration of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ which we will denote by $(F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V))_{i=2}^2$. Namely

$$F_i\mathbf{D}^\dagger_{\mathrm{rig}}(V) \;=\; \mathbf{D}^\dagger_{\mathrm{rig}}(V)\cap (D_i\otimes \mathcal{R}_{\log}\,[1/t]).$$

Following an idea of Greenberg, define

$$W=F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)/F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V).$$

Then we have an exact sequence

$$0\to \mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to W\to \mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to 0 \tag{11}$$

where by Lemma 2.1.5

$$\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V))=\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=p^{-1}},\qquad \mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V))=0,$$
$$\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))=\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=1},\qquad \mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))=\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)).$$

Proposition 2.1.7. *There exists a unique decomposition*

$$W\simeq W_0\oplus W_1\oplus M$$

where W_0 and W_1 are direct summands of $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ and $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ of ranks $\dim_{\mathbb{Q}_p}H^0(W^*(\chi))$ and $\dim_{\mathbb{Q}_p}H^0(W)$ respectively. Moreover, M is inserted in an exact sequence

$$0\to M_0\to M\to M_1\to 0$$

where $\mathrm{gr}_0\mathbf{D}^\dagger_{\mathrm{rig}}(V)\simeq W_0\oplus M_0$, $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\simeq W_1\oplus M_1$ and $\mathrm{rg}(M_0)=\mathrm{rg}(M_1)$.

Proof. Let W_1 denote the saturated (φ,Γ)-submodule of W determined by $\mathcal{D}_{\mathrm{st}}(W_1)=\mathrm{Fil}^0\mathcal{D}_{\mathrm{st}}(W)^{\varphi=1}$ and let N_1 denote the image of $\mathcal{D}_{\mathrm{st}}(W_1)$ in $\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))$. From (11) one has isomorphisms $\mathrm{Fil}^i\mathcal{D}_{\mathrm{st}}(W)\simeq \mathrm{Fil}^i\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ for all $i\geqslant 0$. This implies that $\mathcal{D}_{\mathrm{st}}(W_1)$ and N_1 are isomorphic as filtered Dieudonné modules. Now remark that in the category of filtered vector spaces every subobject is a direct summand. As φ acts trivially on $\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ we obtain that N_1 is a direct summand of $\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ and there exists a projection of $\mathcal{D}_{\mathrm{st}}(W)$ onto $\mathcal{D}_{\mathrm{st}}(W_1)\simeq N_1$. This proves that W_1 is a direct summand of W. Passing to duals and repeating the same arguments we obtain that $W\simeq W_0\oplus W_1\oplus M$. It remains to show that $\mathrm{rg}(M_0)=\mathrm{rg}(M_1)$. As $H^0(M)=0$, we have an exact sequence

$$0\to H^0(M_1)\xrightarrow{\delta} H^1(M_0)\to H^1(M).$$

By 1.5.10 $\dim_{\mathbb{Q}_p}H^1_f(M_1)=\mathrm{rg}M_1$. If $\mathrm{rg}(M_0)<\mathrm{rg}(M_1)$ then there would exist a non zero element $\alpha\in H^0(M_1)$ such that $\delta(\alpha)\in H^1_f(M_0)$. Thus $\delta(\alpha)$ determines a crystalline extension of $\mathcal{R}$ by M_0 which is a subquotient of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. As $M_0\simeq \underset{i}{\oplus}\mathcal{R}(|x|x^{m_i})$ this violates **C4)**. Passing to duals and using the same arguments we obtain the opposite inequality. The proposition is proved.

Lemma 2.1.8. *One has*

$$\dim_{\mathbb{Q}_p} H^1(M)=2r, \quad \dim_{\mathbb{Q}_p} H^1_f(M)=r, \qquad \text{where } r=rg(M_0)=rg(M_1).$$

Moreover $H^1_f(M)=\mathrm{Im}(H^1(M_0)\to H^1(M))$.

Proof. By the definition of W_0 and W_1 we have $H^0(M)=H^0(M^*(\chi))=0$ and the Euler-Poincaré characteristic formula gives $\dim_{\mathbb{Q}_p} H^1(M)=2r$. By Corollary 1.4.5 one has $\dim_{\mathbb{Q}_p} H^1_f(M)=\dim_{\mathbb{Q}_p} t_M=r$. Next, as M has no crystalline subquotient of the form **C4)**, $H^0(M_1)\cap H^1_f(M_0)=\{0\}$. By 1.5.10

$$\dim_{\mathbb{Q}_p} H^0(M_1)+\dim_{\mathbb{Q}_p} H^1_f(M_0)=2r=\dim_{\mathbb{Q}_p} H^1(M_0).$$

Thus $H^1(M_0)\simeq H^0(M_1)\times H^1_f(M_0)$ and the second statement follows from the fact that $\dim_{\mathbb{Q}_p} H^1_f(M_0)=r=\dim_{\mathbb{Q}_p} H^1_f(M)$.

2.1.9. Like in [G], we make the following assumption:

C5) Either W_0 or W_1 is zero.

To fix ideas, assume that $W_0=0$. Otherwise, we consider $V^*(1)$ instead V. Set $r=\mathrm{rg}(M_0)$, $s=\mathrm{rg}(W_1)$ and $e=r+s=\dim_{\mathbb{Q}_p}(D_1/D_0)$. Consider the short exact sequence

$$0\to F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to \mathbf{D}^\dagger_{\mathrm{rig}}(V)\to \mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to 0$$

The quotient $\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is a semistable (φ,Γ)-module and the admissibility of D implies that all the Hodge weights of $\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ are $\geqslant 0$. Moreover $\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\varphi=1,N=0}=0$ and by Proposition 1.4.4 $H^0(\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V))=H^1_f(\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V))=0$. By Corollary 1.4.6 one has an exact sequence

$$0\to H^1_f(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))\to H^1_f(\mathbf{D}^\dagger_{\mathrm{rig}}(V))\to H^1_f(\mathrm{gr}_2\mathbf{D}^\dagger_{\mathrm{rig}}(V))=\{0\}$$

which gives an isomorphism $H^1_f(\mathbb{Q}_p,V)\simeq H^1_f(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))$. Consider the exact sequence

$$0\to F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\to W\to 0.$$

Here $F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is semistable with Hodge weights <0 and $\mathcal{D}_{\mathrm{st}}(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))=D_{-1}$. By **D2)** $\mathrm{Hom}(D_{-1},\mathbb{Q}_p)^{\varphi=1,N=0}=0$ and we obtain

$$\begin{aligned}&H^0(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))=0,\quad H^2(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))\simeq H^0((F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))^*(\chi))^*=0,\\ &H^1_f(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))=H^1(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V)).\end{aligned}$$

Thus $H^1(W)=\mathrm{coker}(H^1(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))\to H^1(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)))$. On the other hand, by Corollary 1.4.6 $H^1_f(W)=\mathrm{coker}(H^1(F_{-1}\mathbf{D}^\dagger_{\mathrm{rig}}(V))\to H^1_f(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)))$ and

$$\frac{H^1(W)}{H^1_f(W)}\simeq \frac{H^1(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))}{H^1_f(\mathbb{Q}_p,V)}.$$

By Lemma 2.1.8 this is a $\mathbb{Q}_p$-vector space of dimension e.

The conditions **C1-2)** together with the exact sequence (9) give an isomorphism

$$H^1_S(V) \simeq \underset{v\in S}{\oplus} \frac{H^1(\mathbb{Q}_v,V)}{H^1_f(\mathbb{Q}_v,V)}.$$

Let $H^1(D,V)$ denote the unique subspace of $H^1_S(V)$ whose image under this map is $H^1(F_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))/H^1_f(\mathbb{Q}_p,V)$. The localization map $H^1_S(V)\to H^1(\mathbb{Q}_p,V)$ gives rise to a commutative diagram

$$\begin{tikzcd}
H^1(D,V) \arrow[d, "\kappa_D"] \arrow[dr, "\bar\kappa_D"] & \\
H^1(W) \arrow[r] & H^1(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))
\end{tikzcd}$$

where κ_D and $\bar\kappa_D$ are injective because by Lemma 2.1.8 $H^1(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \simeq H^1(W)/H^1_f(M)$ and $H^1_f(M)\subset H^1_f(W)$. Hence, we have a diagram

$$\begin{tikzcd}
\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \arrow[r, "i_{D,f}", "\sim"'] & H^1_f(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \\
H^1(D,V) \arrow[u, "\rho_{D,f}"] \arrow[r, "\bar\kappa_D"] \arrow[d, "\rho_{D,c}"'] & H^1(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \arrow[u, "p_{D,f}"'] \arrow[d, "p_{D,c}"] \\
\mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \arrow[r, "i_{D,c}", "\sim"'] & H^1_c(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)),
\end{tikzcd}$$

where $\rho_{D,f}$ and $\rho_{D,c}$ are defined as the unique maps making this diagram commute. From the definition of $H^1(D,V)$ it follows that $\rho_{D,c}$ is an isomorphism.

Definition. *The determinant*

$$\mathcal{L}(V,D)=\det\left(\rho_{D,f}\circ\rho_{D,c}^{-1}\mid \mathcal{D}_{\mathrm{st}}(\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V))\right)$$

will be called the $\mathcal{L}$-invariant associated to V and D.

The isomorphism $\mathrm{gr}_1\mathbf{D}^\dagger_{\mathrm{rig}}(V)\simeq M_1\oplus W_1$ induces a decomposition

$$H^1(D,V)\simeq H^1_{\mathrm{loc}}(D,V)\oplus H^1_{\mathrm{gl}}(D,V)$$

such that $\rho_{D,c}(H^1_{\mathrm{loc}}(D,V))=\mathcal{D}_{\mathrm{st}}(M_1)$ and $\rho_{D,c}(H^1_{\mathrm{gl}}(D,V))=\mathcal{D}_{\mathrm{st}}(W_1)$.

Proposition 2.1.10. *One has*

$$\bar\kappa_D(H^1_{\mathrm{loc}}(D,V))=\mathrm{Im}(H^1(M)\to H^1(M_1)).$$

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(M_1) \arrow[r] & H^1(M_0) \arrow[r] & H^1(M) \arrow[r] & H^1(M_1) \arrow[r] & H^2(M_0) \arrow[r] & 0 \\
& & & H^1_{\mathrm{loc}}(D,V) \arrow[u, "\kappa_D"] \arrow[ur, "\bar\kappa_D"'] & & &
\end{tikzcd}$$

As $H^0(M)=H^2(M)=0$, the upper row is exact. This implies that the image of $H^1(M)$ in $H^1(M_1)$ is a $\mathbb{Q}_p$-vector space of dimension r. On the other hand, $\dim H^1_{\mathrm{loc}}(D,V)=r$ and the proposition follows from the commutativity of the diagram.

Corollary 2.1.11. *One has*

$$\mathcal{L}(V,D)=\mathcal{L}_{\rm loc}(V,D)\,\mathcal{L}_{\rm gl}(V,D)$$

where $\mathcal{L}_{\rm loc}(V,D)=\det\left(\rho_{D,f}\circ\rho_{D,c}^{-1}\mid \mathcal{D}_{\rm st}(M_1)\right)$ *and* $\mathcal{L}_{\rm gl}(V,D)=\det\left(\rho_{D,f}\circ\rho_{D,c}^{-1}\mid \mathcal{D}_{\rm st}(W_1)\right)$. *Here the factor* $\mathcal{L}_{\rm loc}(V,D)$ *is local, i.e. depends only on the restriction of the representation* V *on a decomposition group at* p.

2.2. p-adic L-functions.

2.2.1. Let V be a pseudo-geometric representation which satisfies **C1-5)**. It seems reasonable to expect that to any admissible subspace D of $\mathbf{D}_{\rm st}(V)$ one can associate a meromorphic p-adic L-function $L_p^{\rm an}(V,D,s)$ interpolating special values of the complex L-function $L(V,s)$. A natural generalization of Greenberg's conjecture about the behavior of $L_p^{\rm an}(V,D,s)$ states as follows. If f is a semisimple endomorphism of a vector space W then $W\simeq \ker(f)\oplus \mathrm{Im}(f)$ and we define $\det^*(f)=\det(f\,|\,\mathrm{Im}(f))$.

Greenberg's conjecture. $L_p(V,D,s)$ has a zero of order e at $s=0$. Moreover, $\mathcal{L}(V,D)\neq 0$ and

$$\lim_{s\to 0}\frac{L_p^{\rm an}(V,D,s)}{s^e}=\mathcal{L}(V,D)\,\mathcal{E}(V,D)\,\frac{L(V,0)}{\Omega_\infty(V)}$$

where

$$\mathcal{E}(V,D)=\det{}^*\left(1-p^{-1}\varphi^{-1}\,|D^{N=0}\right)\,\det{}^*\left(1-p^{-1}\varphi^{-1}\,|\,(D^*)^{N=0}\right)$$

and $\Omega_\infty(V)$ denotes the Deligne's period.

Remarks 2.2.2. 1) The interpolation factor $\mathcal{E}(V,D)$ can be written in the form

$$\mathcal{E}(V,D)=\det{}^*\left(1-p^{-1}\varphi^{-1}\,|D^{N=0}\right)\,\det\left(1-\varphi\,|\,\mathbf{D}_{\rm st}(V)/(N\mathbf{D}_{\rm st}(V)+D)\right).$$

Remark that $\mathcal{E}(V,D)=\mathcal{E}(V^*(1),D^*)$.

2) Assume that V is ordinary at p. Then V is equipped with an increasing filtration $\mathcal{F}_iV$ such that $\mathrm{gr}_i(V)\,(-i)$ are unramified. Set $D=\mathbf{D}_{\rm st}(\mathcal{F}_{-1}V)$. Then $\mathcal{L}(V,D)$ coincides with Greenberg's $\mathcal{L}$-invariant and the above conjecture coincides with the conjecture formulated in [G], p. 166.

3) Our definition of the $\mathcal{L}$-invariant generalizes without modifications to the case of an arbitrary coefficient field $L/\mathbb{Q}_p$.

2.2.3. Let V be the p-adic representation associated to a normalized newform f of weight $2k$ on $\Gamma_0(Np)$ ($(N,p)=1$) with Fourier coefficients in $L/\mathbb{Q}_p$. Assume that $U_p(f)=p^{k-1}f$ where U_p is the Atkin-Lehner operator. Then V is semistable of Hodge-Tate weights $(0,2k-1)$ and the (φ,N)-module $\mathbf{D}_{\rm st}(V)$ is generated by two elements d_1 and d_2 such that $\varphi(d_2)=p^kd_2$, $\varphi(d_1)=p^{k-1}d_1$, $Nd_2=d_1$, $Nd_1=0$. The $\mathcal{L}$-invariant of Fontaine-Mazur is defined as the unique element $\mathcal{L}_{\rm FM}(f)\in L$ such that $d_2-\mathcal{L}_{\rm FM}(f)\,d_1\in \mathrm{Fil}^0\mathbf{D}_{\rm st}(V)$.

Then $\mathbf{D}_{\rm st}(V(k))$ is generated by d_1^* and d_2^* such that $\varphi(d_2^*)=d_2^*$, $\varphi(d_1^*)=p^{-1}d_1^*$, $N(d_2^*)=d_1^*$ and $D=Ld_1^*$ is the unique admissible (φ,N)-submodule of $\mathbf{D}_{\rm st}(V(k))$.

Proposition 2.2.4. *One has* $\mathcal{L}_{\rm FM}(f)=\mathcal{L}(V(k),D)$.

Proof. By Proposition 1.3.2 the refinement $\mathcal{F}_0\mathbf{D}_{\rm st}(V)=Ld_1$, $\mathcal{F}^1\mathbf{D}_{\rm st}(V)=\mathbf{D}_{\rm st}(V)$ gives rise to an exact sequence

$$0\to \mathcal{R}_L(|x|^{-(k-1)})\to \mathbf{D}^\dagger_{\rm rig}(V)\to \mathcal{R}_L(x^{-(2k-1)}|x|^{-k})\to 0.$$

We have canonical isomorphisms

$$\mathrm{Ext}^1_{\mathcal{R}_L}(\mathcal{R}_L(x^{-(2k-1)}|x|^{-k}),\mathcal{R}_L(|x|^{-(k-1)}))\simeq H^1(\mathcal{R}_L(x^{2k-1}|x|))\simeq H^1(C^\bullet_{\rm st}(\mathcal{R}_L(x^{2k-1}|x|))).$$

By the proof of Proposition 1.4.4 the class of $\mathbf{D}_{\mathrm{st}}(V)$ in $H^1(C^\bullet_{\mathrm{st}}(\mathcal{R}_L(x^{2k-1}|x|)))$ is $\mathrm{cl}(-\mathcal{L}_{\mathrm{FM}}(f),0,1)$. On the other hand, we have an exact sequence

$$0 \to \mathcal{R}_L(x^k|x|) \to \mathbf{D}^\dagger_{\mathrm{rig}}(V(k)) \to \mathcal{R}_L(x^{1-k}) \to 0.$$

which coincides with the sequence (11) for $V(k)$. Denote by X the image of $H^1(\mathbf{D}^\dagger_{\mathrm{rig}}(V(k)))$ in $H^1(\mathcal{R}_L(x^{-(k-1)}))$. Then $\mathcal{L}(D,V(k))$ is the unique element $\lambda \in L$ such that $y^*_{k-1} - \lambda x^*_{k-1} \in X$. Let $\delta^*_{k-1} : H^0(\mathcal{R}_L(x^{-(k-1)})) \to H^1(\mathcal{R}_L(x^k|x|))$ denote the connecting map associated to the dual exact sequence

$$0 \to \mathcal{R}_L(x^k|x|) \to \mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1-k)) \to \mathcal{R}_L(x^{1-k}) \to 0.$$

By local duality and Proposition 1.5.7 i) $\mathrm{Im}(\delta^*_{k-1})$ is generated by $\beta^*_k - \lambda\alpha^*_k$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}_L(x^{2k-1}|x|) \arrow[r] \arrow[d, "t^{k-1}"] & \mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1-k))(x^{k-1}) \arrow[r] \arrow[d, "t^{k-1}"] & \mathcal{R}_L \arrow[r] \arrow[d, "t^{k-1}"] & 0 \\
0 \arrow[r] & \mathcal{R}_L(x^k|x|) \arrow[r] & \mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1-k)) \arrow[r] & \mathcal{R}_L(x^{1-k}) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are the multiplication by t^{k-1}. This induces a commutative diagram

$$\begin{tikzcd}
H^0(\mathcal{R}_L) \arrow[r, "\delta^*_0"] \arrow[d, "t^{k-1}"] & H^1(\mathcal{R}_L(x^{2k-1}|x|)) \arrow[d, "t^{k-1}"] \\
H^0(\mathcal{R}_L(x^{1-k})) \arrow[r, "\delta^*_{k-1}"] & H^1(\mathcal{R}_L(x^k|x|)).
\end{tikzcd}$$

Together with Corollary 1.5.6 this shows that $\mathrm{Im}(\delta^*_0)$ is generated by $\beta^*_{2k-1} - \lambda\alpha^*_{2k-1}$. On the other hand $\mathrm{Im}(\delta^*_0)$ is generated by the class of $\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1-k))$ in

$$\mathrm{Ext}^1_{\mathcal{R}_L}(\mathcal{R}_L(x^{-(k-1)}), \mathcal{R}_L(x^k|x|)) \simeq H^1(\mathcal{R}_L(x^{2k-1}|x|)).$$

A short direct computation gives $\mathrm{cl}(\mathbf{D}^\dagger_{\mathrm{rig}}(V^*(1-k))) = -\mathrm{cl}(\mathbf{D}^\dagger_{\mathrm{rig}}(V(k)))$. But by Proposition 1.5.8 ii) we have

$$\mathrm{cl}(\mathbf{D}^\dagger_{\mathrm{rig}}(V(k))) = -\beta^*_{2k-1} + \mathcal{L}_{\mathrm{FM}}(f)\,\alpha^*_{2k-1}.$$

This proves that $\lambda = \mathcal{L}_{\mathrm{FM}}(f)$.

2.2.5. Assume that V is crystalline at p. In this case Greenberg's conjecture is compatible with Perrin-Riou's theory of p-adic L-functions. Namely, set $\Gamma_1 = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p(\zeta_p))$ and fix a generator $\gamma_1 \in \Gamma_1$. Let $\Lambda = \mathbb{Z}_p[[\Gamma_1]]$ denote the Iwasawa algebra of Γ_1 and $\mathcal{H}(\Gamma_1) = \{f(\gamma_1 - 1) \mid f \in \mathcal{H}\}$ where $\mathcal{H}$ is the algebra of power series which converges on the open unit disc. Denote by $\mathcal{K}(\Gamma_1)$ the field of fractions of $\mathcal{H}$. Let D be a φ-stable subspace of $\mathbf{D}_{\mathrm{cris}}(V)$ such that $D \oplus \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{cris}}(V)$. In [PR], Perrin-Riou constructed a free Λ-submodule $\mathbf{L}_{\mathrm{Iw}}(V,D)$ of $\mathcal{K}(\Gamma_1)$ in terms of Galois cohomology of V. This construction depends on the choice of a G_S-stable lattice T of V and a $\mathbb{Z}_p$-lattice N of D but for simplicity we do not take it into account here and assume that the p-adic period associated to this choice is equal to 1. The main conjecture of the Iwasawa theory of V states as follow.

Main Conjecture ([PR], [Cz1]). One has $L^{\mathrm{an}}_p(V,D,s) = f(\chi(\gamma_1)^s - 1)$ where f is an appropriate generator of $\mathbf{L}_{\mathrm{Iw}}(V,D)$.

To fix ideas assume that $D^{\varphi=1} = 0$. Then $e = \dim_{\mathbb{Q}_p}(D^{\varphi=p^{-1}})$. In [Ben2] we prove the following result.

Theorem 2.2.6. *Assume that $\mathcal{L}(V,D)\neq 0$. Let f be a generator of $\mathbf{L}_{\mathrm{Iw}}(V,D)$. Then $f(\chi(\gamma_1)^s-1)$ is a meromorphic p-adic function which has a zero at $s=0$ of order e and*

$$\lim_{s\to 0}\frac{f(\chi(\gamma_1)^s-1)}{s^e}\overset{p}{\sim}\mathcal{L}(V,D)\,\mathcal{E}(V,D)\,\frac{\#\text{Ш}(T)}{\#H^0_S(V/T)\,\#H^0_S(V^*(1)/T^*(1))}\,\mathrm{Tam}^0(T)$$

where $\text{Ш}(T)$ is the Tate-Shafarevich group of Bloch and Kato (see [FP]*, section 5.3.4) and $\mathrm{Tam}^0(T)$ is the product of local Tamagawa numbers.*

Remark that the Bloch-Kato conjecture predicts that

$$\frac{L(V,0)}{\Omega_\infty(V)}=\frac{\#\text{Ш}(T)}{\#H^0_S(V/T)\,\#H^0_S(V^*(1)/T^*(1))}\,\mathrm{Tam}^0(T)$$

and Theorem 2.2.6 implies the compatibility of the Greenberg conjecture with Perrin-Riou's theory.

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